Model-based Decision Support 2016

May 12, 2016

Problem Formulation

A repair workshop services two different types of engines M1 and M2. The service of engine M1 lasts 2 days, and the service of M2 one day. Each morning the probability that

- an M1 engine has to be serviced is 1/3,
- an M2 engine has to be serviced is 1/2.

Additional repair orders, which cannot be instantaneously serviced, are serviced elsewhere. Is a working day the first service day of an M1 engine, then all other request on the second day are refused, too. On all other days the management of the repair workshop is free on decision what engine they accept (at each day only one if any),

In general, two different policies are possible:

a) The service of engine M1 is in preference to M2
b) The service of engine M2 is in preference to M1

What policy results in a higher utilization of the repair workshop?

Solution

Compute the transition matrix for 4 different possible states of the system:

- State 0 ... no work
- State 1 ... first service day of an M1 engine
- State 2 ... second service day of an M1 engine
- State 3 ... one day service of an M2 engine

Let us compute the (conditional) probability that there is no work (the system shifts to State 0):

In case that an M1 engine is serviced the first day (state 1), the probability that there is no work on the next day is zero. Otherwise the state no work is independent on the state of the system the day before, and we can compute (IP == probability)

IP (no M1 engine has to be serviced) = 1 – 1/3 = 2/3
IP (no M2 engine has to be serviced) = 1 – 1/2 = 1/2,

Hence the conditional probability that the system is in State 0 (no work) given that in was in State 0,2,3 the day before is:

IP (no work | State 0,2,3) = 2/3 . 1/2 = 1/3.

Applying policy a) we get the full transition matrix P_A (the first column we have just computed):

p_{ij}	0	1	2	3
0	1/3	1/3	0	1/3
1	0	0	1	0
2	1/3	1/3	0	1/3
3	1/3	1/3	0	1/3

Applying policy b) we get the transition matrix P_B (fill in for exercise):

p_{ij}	0	1	2	3
0				
1				
2				
3				

In case of ergodic states (I tell you that above states are ergodic) we know that the steady state probabilities fulfill $\lim_{n\to\infty} p_{ij} = \pi_j$. The steady state probabilities π_j are the solutions of the system of linear equations:

$$\sum_j \pi_j = 1$$

$$\begin{pmatrix}\pi_0 & \pi_1 & \pi_2 & \pi_3\end{pmatrix} \bullet \begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{pmatrix} = \begin{pmatrix}\pi_0 & \pi_1 & \pi_2 & \pi_3\end{pmatrix}$$

where P=(p_{ij}). In case of policy a) we get:

$$1/3 \quad \pi_0 + \quad 0 \quad \pi_1 + \quad 1/3 \quad \pi_2 + \quad 1/3 \quad \pi_3 = \pi_0$$

$$1/3 \quad \pi_0 + \quad 0 \quad \pi_1 + \quad 1/3 \quad \pi_2 + \quad 1/3 \quad \pi_3 = \pi_1$$

$$0 \quad \pi_0 + \quad 1 \quad \pi_1 + \quad 0 \quad \pi_2 + \quad 0 \quad \pi_3 = \pi_2$$

$$1/3 \quad \pi_0 + \quad 0 \quad \pi_1 + \quad 1/3 \quad \pi_2 + \quad 1/3 \quad \pi_3 = \pi_3$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

Verify that the solution of these linear equations is:

$$\pi_0 = 1/4; \pi_1 = 1/4; \pi_2 = 1/4; \pi_3 = 1/4$$

and hence the long run utilization of the repair shop is 75%.

In a similar way in case of applying policy b) we get

$$\ldots \quad \pi_0 + \quad \ldots \quad \pi_1 + \quad \ldots \quad \pi_2 + \quad \ldots \quad \pi_3 = \pi_0$$

$$\ldots \quad \pi_0 + \quad \ldots \quad \pi_1 + \quad \ldots \quad \pi_2 + \quad \ldots \quad \pi_3 = \pi_1$$

$$\ldots \quad \pi_0 + \quad \ldots \quad \pi_1 + \quad \ldots \quad \pi_2 + \quad \ldots \quad \pi_3 = \pi_2$$

$$\ldots \quad \pi_0 + \quad \ldots \quad \pi_1 + \quad \ldots \quad \pi_2 + \quad \ldots \quad \pi_3 = \pi_3$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

verify the steady state probabilities

$$\pi_0 = 2/7; \pi_1 = 1/7; \pi_2 = 1/7; \pi_3 = 3/7$$

Applying policy b) results in a utilization of the repair shop of about 71,5% (exactly 5/7).