Multivariate Statistics: Exercise 1

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Introduction:

1) When $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}^{-1}$ exist, prove each of the following:

- **a)** $(\boldsymbol{A}^\top)^{-1} = (\boldsymbol{A}^{-1})^\top$
- **b)** $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

2) Prove that

$$\boldsymbol{\Gamma A \Gamma}^\top = \sum_{i=1}^{p} a_i \boldsymbol{\gamma}_i \boldsymbol{\gamma}_i^\top \quad ,$$

where $\boldsymbol{\Gamma} = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_p)$, $\boldsymbol{\Gamma}^\top = \boldsymbol{\Gamma}^{-1}$ and $\boldsymbol{A} = Diag(a_1, a_2, \ldots, a_p)$ (cf. lecture notes, Theorem 1.4.3).

3) Show that the matrix

$$\boldsymbol{\Sigma}^{1/2} = \boldsymbol{\Gamma A}^{1/2}\boldsymbol{\Gamma}^\top = \sum_{i=1}^{p} \sqrt{a_i} \boldsymbol{\gamma}_i \boldsymbol{\gamma}_i^\top$$

has the following properties:

- **a)** $(\boldsymbol{\Sigma}^{1/2})^\top = \boldsymbol{\Sigma}^{1/2}$.
- **b)** $\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2} = \boldsymbol{\Sigma}$.
- **c)** $(\boldsymbol{\Sigma}^{1/2})^{-1} = \sum_{i=1}^{p} \frac{1}{\sqrt{a_i}} \boldsymbol{\gamma}_i \boldsymbol{\gamma}_i^\top = \boldsymbol{\Gamma A}^{-1/2}\boldsymbol{\Gamma}^\top$ where $\boldsymbol{A}^{-1/2}$ is a diagonal matrix with $1/\sqrt{a_i}$ as the ith diagonal element.
- **d)** $\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{-1/2} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{1/2} = \boldsymbol{I}$, and $\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{-1/2} = \boldsymbol{\Sigma}^{-1}$, where $\boldsymbol{\Sigma}^{-1/2} = (\boldsymbol{\Sigma}^{1/2})^{-1}$.

4) Show that $Cov(\boldsymbol{Ax}, \boldsymbol{By}) = \boldsymbol{A}\ Cov(\boldsymbol{x}, \boldsymbol{y})\ \boldsymbol{B}^\top$
(cf. lecture notes, Equation (1.22)).

5) Prove: For independent random variables $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^\top$ with $E(x_i) = \mu_i$ and $Var(x_i) = \sigma^2$ for $i = 1, \ldots, n$, and a symmetric matrix $\boldsymbol{A}$ we have:

$$E(\boldsymbol{x}^\top \boldsymbol{Ax}) = \boldsymbol{\mu}^\top \boldsymbol{A\mu} + \sigma^2\ tr(\boldsymbol{A}) \quad ,$$

where $\boldsymbol{\mu}^\top = (\mu_1, \mu_2, \ldots, \mu_n)$, and tr denotes the trace of a matrix.