2. Übung Höhere Wahrscheinlichkeitstheorie

Fun with topology and stuff

The right limit topology on the real line is not locally compact (a compact set must not contain an inffinite increasing sequence, so for any $x \in K$ there is an interval $(x-\epsilon_x, x)$ that does not intersect K, so K is at most countable and well-ordered with $\geq$).

What are the compact sets in the rational sequence topology?

On any set X, there is a weakest $T1$ topology.

A topological space X is Hausdorff iff the diagonal $\{(x,x), x \in X\}$ is closed in the product topology.

A topological space is Hausdorff iff every convergent filter has a unique limit. This statement is not true for sequences (give an example), but $T2$ implies unique sequence limits implies $T1$. Give a counterexample showing that $T1$ in general does not imply unique sequence limits.

Every closed subset of a compact space is compact. If the space is also Hausdorff, the reverse is also true; in addition, a compact Hausdorff space is normal and of category 2.

A compact metric space X is complete and totally bounded (aka precompact): for every $\epsilon > 0$ there is a finite cover by balls of radius ϵ. In reverse, a metric space is compact if it is complete and totally bounded.

Let X be any set. For $A \subseteq X \times X$, let $A(x_1, .) = \{x_2 : (x_1, x_2) \in A\}$ denote the x_1-section of A.

$$\mathfrak{A} = \{A \subseteq X \times X : |\{A(x_1, .), x \in X\}| \leq |\mathbb{R}|\}$$

is a sigma-algebra.

For any topological space, $\mathfrak{B}(X) \times \mathfrak{B}(X) \subseteq \mathfrak{B}(X \times X)$. In general, this inclusion is strict.

From now on, we tacitly assume that all topologies are Hausdorff.

A locally compact topology is completely regular.

Counting measure is locally finite iff the underlying topology is discrete. When is it regular from above resp. below?

An arbitrary sum of measures that are regular from below is also regular from below.

The one-point or Alexandrov compactification of a space $(X, \mathcal{T})$ is given by $X' = X \cup \{\infty\}$ and $\mathcal{T}' = \mathcal{T} \cup \{\{\infty\} \cup K^C, K \subset X \text{ compact}\}$. This is indeed compact but not necessarily Hausdorff. When is it?

In a first countable space, countable compactness implies seqquential compactness, in a second countable space compactness.