3. Übung Höhere Wahrscheinlichkeitstheorie

1. We call a measure μ on $\mathfrak{B}(X)$ tight if for any $\epsilon > 0$ there is a compact set K with $\mu(K^C) < \epsilon$).

   Show that if a finite measure μ is regular from below, it is also regular from above; the converse holds true iff μ is tight.

2. Let $(X, \mathcal{T})$ be a topological space; we call a function $f : \mathcal{T} \to \mathbb{R}$ $\mathcal{T}$-smooth if for any directed subset of $\mathcal{U} \subset \mathcal{T}$ (i.e., the elements of $\mathcal{U}$ are open sets, an for $U1, U2 \in \mathcal{U}$ there is a $V \in \mathcal{U}$ with $U_1 \cup U_2 \subseteq V$)

$$\mu(\bigcup_{U \in \mathcal{U}} U) = \sup\{\mu(U), u \in \mathcal{U}\}.$$

   Show that every measure that is regular from below is $\mathcal{T}$-smooth.

3. A finitely additive set function on $\mathfrak{B}(X)$ that is $\mathcal{T}$-smooth is sigma-additive.

4. For a measure μ on $\mathfrak{B}(X)$, let $\mathfrak{F}_\mu$ be the set of all $A \in \mathfrak{B}(X)$ for which one can find, for any $epsilon > 0$, an open set U and a closed set C with $C \subseteq A \subseteq U$ and $\mu(U \setminus C) < \epsilon$. Show that F_μ is a sigmaalgebra.

5. Call a set $A \in \mathfrak{B}(X)$ regular from below (above) if $\mu(A) = \sup\{\mu(C) : C \subseteq A, C \text{ compact}\}$ ($\mu(A) = \inf\{\mu(U) : A \subseteq U, U \text{ open}\}$).

   A finite measure is regular if all open sets are regular from below (use the previous problem and problem 1).