Exercise 8

In this Exercise we will study the Brownian Motion and relate it to the concepts of Markov properties and Feller processes which we have discussed in class[1].

1 Working on the space $(\Omega, \mathcal{F})$ (where $\Omega := \{\omega : [0, \infty) \to \mathbb{R} : \omega \text{ is càdlàg}\}$ and $\mathcal{F}$ the smallest σ-algebra making $\omega \mapsto \omega(t)$ measurable for each $t \geq 0$), specify the properties required of a measure $\mathbb{P}_x$ on this space for the canonical process $X_t(\omega) = \omega(t)$ to be a Brownian motion started in x (i.e. effectively just give the definition of Brownian motion).

2 Show that $(\mathbb{P}_x)_{x\in\mathbb{R}}$ given as in [1] (i.e. being the Brownian motion) satisfies the Feller property.

3 Consider the following two filtrations: the (raw) filtration $\mathbb{F}^0 = (\mathcal{F}_t^0)_{t\geq 0}$ where $\mathcal{F}_t^0 = \sigma(X_s : s \leq t)$ and the (right-continuous) filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ where $\mathcal{F}_t = \cap_{\varepsilon>0}\mathcal{F}^0_{t+\varepsilon}$. Consider now the 'first exit time': $\tau(\omega) = \inf\{t \geq 0 : X_t > c\}$, for some $c > 0$. Argue whether it is a stopping time with respect to both $\mathbb{F}^0$ and $\mathbb{F}$, only either $\mathbb{F}^0$ or $\mathbb{F}$ (which one), or none of them.

4 Argue that (the Brownian motion) $(\mathbb{P}_x)_{x\in\mathbb{R}}$ (as given in [1]) satisfies the Markov Property (MP) with respect to the filtration $\mathbb{F}^0$.

5 Argue that (the Brownian motion) $(\mathbb{P}_x)_{x\in\mathbb{R}}$ (as given in [1]) satisfies the Markov Property (MP) with respect to the filtration $\mathbb{F}$.

Hints:

- Argue that w.l.o.g. it suffices to consider random variables Z of the form
$$Z = \Pi_{i=1}^n f_i(X_{t_i}), \quad 0 < t_1 < ... < t_n, \quad f_i \in C(\mathbb{R}).$$
- For any $h > 0$, introduce a random variable Z^h such that $Z = Z^h \circ \theta_h$. In turn, applying the MP from [4] to Z^h (conditioning on $\mathcal{F}^0_{s+h}$), deduce that for any $A \in \mathcal{F}_s$
$$\mathbb{E}_x\left[\mathbf{1}_A Z \circ \theta_s\right] = \mathbb{E}_x\left[\mathbf{1}_A \mathbb{E}_{X_{s+h}}\left[Z^h\right]\right].$$
- Conclude by verifying that one may pass to the limit on the r.h.s. in the above equality.

In summary, we have shown that (the Brownian Motion) $((\mathbb{P}_x)_{x\in\mathbb{R}}, (\mathcal{F}_t)_{t\geq 0})$ is a Feller process. In particular, it thus has the Strong Markov Property (SMP) and in consequence e.g. the reflection principle holds.

[1] If you follow the class, please use the definitions provided there. If you do not follow the class, please google for appropriate definitions; if you do not manage to find this, please contact me.