1. Test - Lösungen **9.12.2011**

1 Indexschreibweise

$\text{rot}\,(\mathbf{F}\times\mathbf{x}) \to \varepsilon_{ijk}\partial_j\left(\varepsilon_{klm}F_l x_m\right)$
$= \underbrace{\varepsilon_{ijk}\varepsilon_{klm}}_{\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}} \underbrace{\partial_j\left(F_l x_m\right)}_{(\partial_j F_l)x_m + F_l \underbrace{(\partial_j x_m)}_{\delta_{jm}}}$
$= \left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\left[\left(\partial_j F_l\right)x_m + F_l\delta_{jm}\right]$
$= \delta_{il}\delta_{jm}\left(\partial_j F_l\right)x_m + \delta_{il}\delta_{jm}F_l\delta_{jm} - \delta_{im}\delta_{jl}\left(\partial_j F_l\right)x_m - \delta_{im}\delta_{jl}F_l\delta_{jm}$
$= x_j\left(\partial_j F_i\right) + \underbrace{\delta_{jj}}_{3}F_i - \left(\partial_j F_j\right)x_i - F_i$
$= x_j\left(\partial_j F_i\right) - x_i\left(\partial_j F_j\right) + 2F_i$
$\to \left(\mathbf{x}\cdot\nabla\right)\mathbf{F} - \mathbf{x}\left(\nabla\cdot\mathbf{F}\right) + 2\mathbf{F}$

2 Delta-Distribution

$I = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\,\delta(xy+2y)\underbrace{\delta(2x-2y-4)}_{\frac{1}{2}\delta(x-y-2)}f(x,y)$

Integration über x: $x = y+2$

$= \frac{1}{2}\int_{-\infty}^{\infty} dy\,\delta(\underbrace{(y+2)\,y+2y}_{y^2+4y=y(y+4)})f(y+2,y)$

Ableitung: $\left(y^2+4y\right)' = 2y+4$

$= \frac{1}{2}\int_{-\infty}^{\infty} dy\left[\underbrace{\frac{1}{|2y+4|}\delta\left(y\right)}_{\frac{1}{4}\delta(y)} + \underbrace{\frac{1}{|2y+4|}\delta(y+4)}_{\frac{1}{4}\delta(y+4)}\right]f(y+2,y)$

$= \frac{1}{8}\left[f(2,0)+f(-2,-4)\right]$

3 Tensoren

a) Transformationsmatrix $\mathbf{e}''_i = {a_i}^j \mathbf{e}_j$: ${a_1}^1 = 1$, ${a_1}^2 = 0$, ${a_2}^1 = 2$, ${a_2}^2 = 1$.
$A'_{jk} = {a_j}^l {a_k}^m A_{lm}$.
$A'_{11} = {a_1}^l {a_1}^m A_{lm} = {a_1}^1 {a_1}^1 A_{11} + {a_1}^1 {a_1}^2 A_{12} + {a_1}^2 {a_1}^1 A_{21} + {a_1}^2 {a_1}^2 A_{22} = 0 + 0 + 0 + 0 = 0.$
$A'_{12} = {a_1}^l {a_2}^m A_{lm} = {a_1}^1 {a_2}^1 A_{11} + {a_1}^1 {a_2}^2 A_{12} + {a_1}^2 {a_2}^1 A_{21} + {a_1}^2 {a_2}^2 A_{22} = 0 + 1 + 0 + 0 = 1.$
$A'_{21} = {a_2}^l {a_1}^m A_{lm} = {a_2}^1 {a_1}^1 A_{11} + {a_2}^1 {a_1}^2 A_{12} + {a_2}^2 {a_1}^1 A_{21} + {a_2}^2 {a_1}^2 A_{22} = 0 + 0 + 0 + 0 = 0.$
$A'_{22} = {a_2}^l {a_2}^m A_{lm} = {a_2}^1 {a_2}^1 A_{11} + {a_2}^1 {a_2}^2 A_{12} + {a_2}^2 {a_2}^1 A_{21} + {a_2}^2 {a_2}^2 A_{22} = 0 + 2 + 0 + 0 = 2.$
b) $g_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j$.

$$g'_{11} = \mathbf{e}'_1 \cdot \mathbf{e}'_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1.$$

$$g'_{12} = \mathbf{e}'_1 \cdot \mathbf{e}'_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2.$$

$$g'_{21} = \mathbf{e}'_2 \cdot \mathbf{e}'_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2.$$

$$g'_{22} = \mathbf{e}'_2 \cdot \mathbf{e}'_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5.$$

c) $g''^{ij} g''_{jk} = \delta^i_k$
$\rightarrow$Inverse bilden:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 5 & -2 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

$$g'^{ij} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}.$$

d) $s = g'^{ij} A'_{jk} g'^{km} \left(A'_{mi} - A'_{im}\right)$
$= g^{ij} A_{jk} g^{km} \left(A_{mi} - A_{im}\right)$
$= A^{im} A_{mi} - A^{im} A_{im}$

$$= \mathrm{tr} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \mathrm{tr} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \mathrm{tr} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \mathrm{tr} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$= 0 - 1 = -1.$