**2. Test - Lösungen** **20.01.2012**

# 1 Separationsansatz

$\left(y\frac{\partial^2}{\partial x^2}+\frac{1}{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\Phi(x,y,z)=\left(\lambda+y^2\right)\Phi(x,y,z)$
Ansatz: $\Phi(x,y,z)=\Phi_1(x)\Phi_2(y)\Phi_3(z)$.
Ganze Gleichung durch $\Phi=\Phi_1\Phi_2\Phi_3$ dividieren:
$\frac{1}{\Phi_1}\left[y\frac{\partial^2}{\partial x^2}\Phi_1\right]+\frac{1}{\Phi_2}\left[\frac{1}{y}\frac{\partial}{\partial y}\Phi_2\right]+\frac{1}{\Phi_3}\left[\frac{\partial}{\partial z}\Phi_3\right]=\lambda+y^2$.
$\Phi_3$ abspalten:
$\frac{1}{\Phi_1}\left[y\frac{\partial^2}{\partial x^2}\Phi_1\right]+\frac{1}{\Phi_2}\left[\frac{1}{y}\frac{\partial}{\partial y}\Phi_2\right]-y^2=-\frac{1}{\Phi_3}\left[\frac{\partial}{\partial z}\Phi_3\right]+\lambda=A(x,y)=A(z)=A=const.$
$\to\Phi_3'(z)=(\lambda-A)\,\Phi_3(z)$.
Durch $y$ dividieren, $\Phi_2$ abspalten:
$\frac{1}{\Phi_1}\left[\frac{\partial^2}{\partial x^2}\Phi_1\right]=\frac{A}{y}-\frac{1}{\Phi_2}\left[\frac{1}{y^2}\frac{\partial}{\partial y}\Phi_2\right]+y=B(x)=B(y)=B=const.$
$\to\frac{\partial^2}{\partial x^2}\Phi_1(x)=B\Phi_1(x)$
$\to\frac{\partial}{\partial y}\Phi_2(y)=\left(Ay+y^3-By^2\right)\Phi_2(y)$

# 2 Sturm-Liouville-Problem

a) $a_0(x)=e^{2x}$, $a_1(x)=-e^{-2x}$, $a_2(x)=-e^{-2x}$.
$p(x)=e^{\int\frac{a_1(x)}{a_2(x)}dx}=\exp\left(\int\frac{-e^{-2x}}{-e^{-2x}}dx\right)=\exp\left(\int 1dx\right)=\exp(x+c)=\tilde{c}e^x$.
$q(x)=p(x)\frac{a_0(x)}{a_2(x)}=\tilde{c}e^x\frac{e^{2x}}{-e^{-2x}}=-\tilde{c}e^{5x}$.
Sturm-Liouville Form:
$\mathcal{S}_x y(x)=\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]y(x)+q(x)y(x)=\frac{d}{dx}\left[e^x\frac{d}{dx}\right]y(x)-e^{5x}y(x)=(e^xy')'-e^{5x}y$.
b) $\rho(x)=-\frac{p(x)}{a_2(x)}=-\frac{\tilde{c}e^x}{-e^{-2x}}=\tilde{c}e^{3x}$.
Sturm-Liouville Transformation:
$t(x)=\int_{-\infty}^{x}\sqrt{\frac{\rho(s)}{p(s)}}ds=\int_{-\infty}^{x}\sqrt{\frac{\tilde{c}e^{3s}}{\tilde{c}e^{s}}}ds=\int_{-\infty}^{x}e^sds=e^x-e^{-\infty}=e^x\quad\to\quad x(t)=\ln t$.
$\hat{q}(t)=\frac{1}{\tilde{c}e^{3x}}\left[\tilde{c}e^{5x}-\sqrt[4]{\tilde{c}e^x\tilde{c}e^{3x}}\left(\tilde{c}e^x\left(\frac{1}{\sqrt{\tilde{c}e^x}}\right)'\right)'\right]$
$=\frac{1}{\tilde{c}e^{3x}}\left[\tilde{c}e^{5x}-\sqrt{\tilde{c}}e^x\left(\tilde{c}e^x\left(\frac{1}{\sqrt{\tilde{c}}}e^{-x}(-1)\right)\right)'\right]=\frac{1}{\tilde{c}e^{3x}}\left[\tilde{c}e^{5x}-\sqrt{\tilde{c}}e^x\left(-\sqrt{\tilde{c}}\right)'\right]$
$=\frac{1}{\tilde{c}e^{3x}}\left[\tilde{c}e^{5x}-0\right]=e^{2x}=e^{2\ln t}=t^2$
Grenzen: $t(0)=e^{-\infty}=0$, $t(b)=e^b$.
Liouvillesche Normalform:
$\frac{d^2}{dt^2}w(t)+\left[t^2-\lambda\right]w(t)=0$ für $t\in[0,e^b]$.
Alternative:
$H(x)=\frac{1}{\sqrt[4]{\tilde{c}e^x\tilde{c}e^{3x}}}=\frac{1}{\sqrt{\tilde{c}}e^x}$, $H'(x)=-\frac{1}{\sqrt{\tilde{c}}e^x}$, $H''(x)=\frac{1}{\sqrt{\tilde{c}}e^x}$
$\hat{q}(t)=\frac{1}{H}\mathcal{L}_xH=\sqrt{\tilde{c}}e^x\left(-\frac{H''}{e^{2x}}-\frac{H'}{e^{2x}}+e^{2x}H\right)=e^x\left(-\frac{1}{e^{3x}}+\frac{1}{e^{3x}}+e^x\right)=e^{2x}=t^2$.

# 3 Greensche Funktion

a) $\mathcal{L}_x = -\frac{d^2}{dx^2} + 1$, $f(x) = 2$, $\mathcal{L}_x y(x) = f(x)$.
Ansatz: $G(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k) e^{ik(x-x')} dk$
und $\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')} dk$ einsetzen in
$\mathcal{L}_x G(x,x') = \delta(x-x')$,
$\left(-\frac{d^2}{dx^2} + 1\right) G(x,x') = \delta(x-x')$,
$\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k) \left(-\frac{d^2}{dx^2} + 1\right) e^{ik(x-x')} dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')} dk$,
$\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k) \left(k^2 + 1\right) e^{ik(x-x')} dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')} dk$.
Vergleich der Integranden:
$\tilde{G}(k)\left(k^2+1\right) = 1. \quad \to \quad \tilde{G}(k) = \frac{1}{k^2+1}$.
b)
$G(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik(x-x')}}{k^2+1} dk$.
Pole liegen bei $k = \pm i$. Für $x - x' > 0$: Großkreis oben schließen ($ik = i(\text{Re}k + i\text{Im}k) = i\text{Re}k - \text{Im}k$: Für $\text{Im}k > 0$ exponentiell gedämpft). Für $x - x' < 0$: Großkreis unten schließen $\to$ Vorzeichen.
$G(x,x') = \theta(x-x') 2\pi i \text{Res}_{k\to i} \frac{1}{2\pi}\frac{e^{ik(x-x')}}{k^2+1} - \theta(x'-x) 2\pi i \text{Res}_{k\to -i} \frac{1}{2\pi}\frac{e^{ik(x-x')}}{k^2+1}$
$= \theta(x-x') 2\pi i \lim_{k\to i} (k-i) \frac{1}{2\pi}\frac{e^{ik(x-x')}}{(k+i)(k-i)} - \theta(x'-x) 2\pi i \lim_{k\to -i} (k+i) \frac{1}{2\pi}\frac{e^{ik(x-x')}}{(k+i)(k-i)}$
$= \theta(x-x') i \frac{e^{-(x-x')}}{2i} - \theta(x'-x) i \frac{e^{x-x'}}{-2i}$
$= \theta(x-x') \frac{e^{-(x-x')}}{2} + \theta(x'-x) \frac{e^{x-x'}}{2}$.
c)
Randbedingung: $G(0, x' > 0) = \frac{e^{x-x'}}{2}$ nicht erfüllt.
Homogene Greensche Funktion über Ansatz:
$G = G_I + Ae^{-x+x'} + Be^{x-x'}$.
$G(0, x' > 0) = \frac{e^{-x'}}{2} + Ae^{x'} + Be^{-x'} = 0$. (I)
$G'(x,x') = \frac{1}{2}\left[\delta(x-x')e^{-x+x'} - \theta(x-x')e^{-x+x'} + (-1)\delta(x'-x)e^{x-x'} + \theta(x'-x)e^{x-x'}\right]$
$-Ae^{-x+x'} + Be^{x-x'}$.
$G'(0, x' > 0) = \frac{1}{2}e^{-x'} - Ae^{x'} + Be^{-x'} = 0$. (II)
Aus (I) und (II) folgt: $A = 0$, $B = -\frac{1}{2}$.
$\to G(x,x') = G_I - \frac{1}{2}e^{x-x'} = \theta(x-x')\frac{e^{-(x-x')}}{2} + \theta(x'-x)\frac{e^{x-x'}}{2} - \frac{1}{2}e^{x-x'}$
$= \frac{1}{2}\theta(x-x')\left[e^{-(x-x')} - e^{x-x'}\right]$.
d)
Lösung: $y(x) = \int_0^{\infty} G(x,x') f(x') dx' = \int_0^x dx' \frac{e^{-(x-x')}}{2} 2 + \int_x^{\infty} dx' \frac{e^{x-x'}}{2} 2 - \int_0^{\infty} dx' \frac{e^{x-x'}}{2} 2$
$= e^{-x+x'}\Big|_{x'=0}^{x} + \frac{e^{x-x'}}{-1}\Big|_{x'=x}^{\infty} - \frac{e^{x-x'}}{-1}\Big|_{x'=0}^{\infty}$
$= 1 - e^{-x} - e^{-\infty} + 1 + e^{-\infty} - e^{x}$
$= 2 - e^{-x} - e^{x}$.