5. Tutorium - Lösungen **18.11.2011**

5.1 Differentialoperatoren

a) $\operatorname{div}\operatorname{rot}\mathbf{v} \to \partial_i \varepsilon_{ijk}\partial_j v_k = \underbrace{\varepsilon_{ijk}}_{\text{antisymmetrisch}} \underbrace{\partial_i\partial_j}_{\text{symmetrisch}} v_k = 0.$

b) $\operatorname{rot}\operatorname{grad}\varphi \to \underbrace{\varepsilon_{ijk}}_{\text{antisymmetrisch}} \underbrace{\partial_j\partial_k}_{\text{symmetrisch}} \varphi = 0.$

c) $\operatorname{rot}\operatorname{rot}\mathbf{v} \to \varepsilon_{ijk}\partial_j\varepsilon_{klm}\partial_l v_m = \underbrace{\varepsilon_{ijk}\varepsilon_{klm}}_{\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}} \partial_j\partial_l v_m = \underbrace{\partial_j\partial_i v_j}_{\partial_i\partial_j v_j} - \partial_j\partial_j v_i \to \operatorname{grad}\operatorname{div}\mathbf{v} - \Delta\mathbf{v}.$

d) $\mathbf{A}\cdot[\nabla\times(\nabla\times\mathbf{A}) - \nabla(\nabla\cdot\mathbf{A})] \to A_i\left[\varepsilon_{ijk}\partial_j\left(\varepsilon_{klm}\partial_l A_m\right) - \partial_i\left(\partial_j A_j\right)\right] = A_i\left[\underbrace{\varepsilon_{ijk}\varepsilon_{klm}}_{\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}} \partial_j\partial_l A_m - \partial_i\partial_j A_j\right]$

$= A_i\left[\underbrace{\partial_j\partial_i A_j}_{\partial_i\partial_j A_j} - \partial_j\partial_j A_i - \partial_i\partial_j A_j\right] = -A_i\left(\partial_j\partial_j A_i\right) \to -\mathbf{A}\left(\Delta\mathbf{A}\right).$

e) $\nabla\cdot\mathbf{x} \to \partial_i x_i = \delta_{ii} = 3.$

f) $\nabla r \to \partial_i r = \partial_i\sqrt{x_j x_j} = \frac{1}{2}\frac{1}{\sqrt{x_j x_j}}\left[\underbrace{\left(\partial_i x_j\right)}_{\delta_{ij}} x_j + x_j\underbrace{\left(\partial_i x_j\right)}_{\delta_{ij}}\right] = \frac{2\delta_{ij}x_j}{2r} = \frac{x_i}{r}.$

g) $\nabla\cdot\left(\frac{\mathbf{x}}{r^5}\right) \to \partial_i\left(\frac{x_i}{r^5}\right) = \frac{1}{r^5}\underbrace{\left(\partial_i x_i\right)}_{\delta_{ii}=3} + x_i\left(\partial_i\frac{1}{r^5}\right) = \frac{1}{r^5}3 - x_i 5\frac{1}{r^6}\left(\partial_i r\right) = \frac{3}{r^5} - x_i\frac{5}{r^6}\frac{x_i}{r} = -\frac{2}{r^5}$ (mit $x_i x_i = r^2$).

h) $\nabla\left(\frac{\mathbf{p}\cdot\mathbf{x}}{r^5}\right) \to \partial_i\left(\frac{p_j x_j}{r^5}\right) = p_j\frac{1}{r^5}\underbrace{\left(\partial_i x_j\right)}_{\delta_{ij}} + p_j x_j\Big(\underbrace{\partial_i\frac{1}{r^5}}_{-\frac{5}{r^6}\partial_i r}\Big) = p_j\frac{1}{r^5}\delta_{ij} - p_j x_j\frac{5}{r^6}\frac{x_i}{r} = \frac{p_i}{r^5} - \frac{5x_i p_j x_j}{r^7} \to \frac{\mathbf{p}}{r^5} - \frac{5\mathbf{x}(\mathbf{p}\cdot\mathbf{x})}{r^7}.$

5.2 Tensorfelder

a) $Q(x) = \begin{pmatrix} x_1^2 & 0 \\ 0 & x_2^2 \end{pmatrix}.$

„Äußere“ Transformation: $\tilde{Q}'_{ij}(x_n) = a_{ik}a_{jl}Q_{kl}(x_n) \to \tilde{Q}'(\mathbf{x}) = a\cdot Q(\mathbf{x})\cdot a^T$ mit $a = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$

$\tilde{Q}'(\mathbf{x}) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} x_1^2 & 0 \\ 0 & x_2^2 \end{pmatrix}\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi x_1^2 & \sin\varphi x_2^2 \\ -\sin\varphi x_1^2 & \cos\varphi x_2^2 \end{pmatrix}\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

$= \begin{pmatrix} c^2x_1^2 + s^2x_2^2 & -csx_1^2 + csx_2^2 \\ -csx_1^2 + csx_2^2 & s^2x_1^2 + c^2x_2^2 \end{pmatrix}$

$(c \equiv \cos\varphi,\ s \equiv \sin\varphi)$

„Innere“ Transformation: $x'_i = a_{ij}x_j$, demnach $x_i = \left(a^{-1}\right)_{ij}x'_j$.

$Q'_{ij}(x'_n) = \tilde{Q}'_{ij}\left(\left(a^{-1}\right)_{ij}x'_j\right).$

Zur „Überprüfung“ kann man aber einfach $x'_i = a_{ij}x_j$ in die ursprüngliche Form einsetzen.

$Q'(\mathbf{x}') \stackrel{?}{=} \begin{pmatrix} x'^2_1 & 0 \\ 0 & x'^2_2 \end{pmatrix} = \begin{pmatrix} \left(\cos\varphi x_1 + \sin\varphi x_2\right)^2 & 0 \\ 0 & \left(-\sin\varphi x_1 + \cos\varphi x_2\right)^2 \end{pmatrix}$

$= \begin{pmatrix} c^2x_1^2 + 2csx_1x_2 + s^2x_1^2 & 0 \\ 0 & c^2x_1^2 - 2csx_1x_2 + s^2x_1^2 \end{pmatrix} = Q'(x).$

Nachdem $\tilde{Q}'(x)$ nicht mit $Q'(x)$ übereinstimmt, ist dieser Tensor nicht forminvariant.

b) $R(x) = \begin{pmatrix} x_1^2 + x_2^2 & 0 \\ 0 & x_1^2 + x_2^2 \end{pmatrix}.$

„Äußere“ Transformation: $\tilde{R}'_{ij}(x_n) = a_{ik}a_{jl}R_{kl}(x_n) \to \tilde{R}'(\mathbf{x}) = a \cdot R(\mathbf{x}) \cdot a^T$ mit $a = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$.

$\tilde{R}'(\mathbf{x}) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x_1^2 + x_2^2 & 0 \\ 0 & x_1^2 + x_2^2 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \left(x_1^2 + x_2^2\right) \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$

$= \left(x_1^2 + x_2^2\right) \begin{pmatrix} c^2 + s^2 & -cs + cs \\ -cs + cs & s^2 + c^2 \end{pmatrix} = \left(x_1^2 + x_2^2\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

„Innere“ Transformation:

Zur „Überprüfung“ kann man $x'_i = a_{ij}x_j$ in die ursprüngliche Form einsetzen:

$R'(\mathbf{x}') \stackrel{?}{=} \begin{pmatrix} {x'}_1^2 + {x'}_2^2 & 0 \\ 0 & {x'}_1^2 + {x'}_2^2 \end{pmatrix} = \left((cx_1 + sx_2)^2 + (-sx_1 + cx_2)^2\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \left[\left(c^2 + s^2\right) x_1^2 + \left(c^2 + s^2\right) x_2^2 + (2sc - 2sc)\, x_1 x_2\right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(x_1^2 + x_2^2\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = R'(x)$.

Da $\tilde{R}'(x)$ und $R'(x)$ übereinstimmen, ist dieser Tensor forminvariant.

c) $S(x) = \begin{pmatrix} 0 & x_1^2 + x_2^2 \\ x_1^2 + x_2^2 & 0 \end{pmatrix}$.

„Äußere“ Transformation:

$\tilde{S}'(\mathbf{x}) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 0 & x_1^2 + x_2^2 \\ x_1^2 + x_2^2 & 0 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \left(x_1^2 + x_2^2\right) \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$

$= \left(x_1^2 + x_2^2\right) \begin{pmatrix} s & c \\ c & -s \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix} = \left(x_1^2 + x_2^2\right) \begin{pmatrix} 2cs & c^2 - s^2 \\ c^2 - s^2 & -2cs \end{pmatrix}$.

„Innere“ Transformation:

$S'(\mathbf{x}') \stackrel{?}{=} \begin{pmatrix} 0 & {x'}_1^2 + {x'}_2^2 \\ {x'}_1^2 + {x'}_2^2 & 0 \end{pmatrix} = \left(x_1^2 + x_2^2\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = S'(x)$.

Da $\tilde{S}'(x)$ und $S'(x)$ nicht übereinstimmen, ist dieser Tensor nicht forminvariant.

d) $T(x) = \begin{pmatrix} x_1^2 - 1 & x_1 x_2 - 1 \\ x_1 x_2 + 1 & x_2^2 - 1 \end{pmatrix}$.

„Äußere“ Transformation:

$\tilde{T}'(\mathbf{x}) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x_1^2 - 1 & x_1 x_2 - 1 \\ x_1 x_2 + 1 & x_2^2 - 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

$= \begin{pmatrix} c\left(x_1^2 - 1\right) + s\,(x_1 x_2 + 1) & c\,(x_1 x_2 - 1) + s\left(x_2^2 - 1\right) \\ -s\left(x_1^2 - 1\right) + c\,(x_1 x_2 + 1) & -s\,(x_1 x_2 - 1) + c\left(x_2^2 - 1\right) \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$

$= \begin{pmatrix} c^2\left(x_1^2 - 1\right) + cs\,(x_1 x_2 + 1) + cs\,(x_1 x_2 - 1) + s^2\left(x_2^2 - 1\right) & -cs\left(x_1^2 - 1\right) - s^2\,(x_1 x_2 + 1) + c^2\,(x_1 x_2 - 1) + cs\left(x_2^2 - 1\right) \\ -cs\left(x_1^2 - 1\right) + c^2\,(x_1 x_2 + 1) - s^2\,(x_1 x_2 - 1) + cs\left(x_2^2 - 1\right) & s^2\left(x_1^2 - 1\right) - cs\,(x_1 x_2 + 1) - cs\,(x_1 x_2 - 1) + c^2\left(x_2^2 - 1\right) \end{pmatrix}$

$= \begin{pmatrix} c^2 x_1^2 + 2cs x_1 x_2 + s^2 x_2^2 - 1 & -cs x_1^2 + \left(c^2 - s^2\right) x_1 x_2 + cs x_2^2 - 1 \\ -cs x_1^2 + \left(c^2 - s^2\right) x_1 x_2 + cs x_2^2 + 1 & s^2 x_1^2 - 2cs x_1 x_2 + c^2 x_2^2 - 1 \end{pmatrix}$.

„Innere“ Transformation:

$T'(\mathbf{x}') \stackrel{?}{=} \begin{pmatrix} {x'}_1^2 - 1 & x'_1 x'_2 - 1 \\ x'_1 x'_2 + 1 & {x'}_2^2 - 1 \end{pmatrix}$

$= \begin{pmatrix} (cx_1 + sx_2)^2 - 1 & (cx_1 + sx_2)(-sx_1 + cx_2) - 1 \\ (cx_1 + sx_2)(-sx_1 + cx_2) + 1 & (-sx_1 + cx_2)^2 - 1 \end{pmatrix}$

$= \begin{pmatrix} c^2 x_1^2 + 2cs x_1 x_2 + s^2 x_2^2 - 1 & (\cos\varphi x_1 + \sin\varphi x_2)(-\sin\varphi x_1 + \cos\varphi x_2) - 1 \\ -cs x_1^2 + \left(c^2 - s^2\right) x_1 x_2 + cs x_2^2 + 1 & s^2 x_1^2 - 2cs x_1 x_2 + c^2 x_2^2 - 1 \end{pmatrix}$

$= T'(x)$

Da $\tilde{T}'(x)$ und $T'(x)$ übereinstimmen, ist dieser Tensor forminvariant.

5.3 Kugelkoordinaten

a) $g'_{ij} = \frac{\partial x^l}{\partial x'^i} \frac{\partial x_l}{\partial x'^j}$. Mit $x'^1 = r$, $x'^2 = \theta$, und $x'^3 = \varphi$ ergibt sich

$g'_{11} = \frac{\partial x^l}{\partial x'^1} \frac{\partial x_l}{\partial x'^1} = \frac{\partial(r \sin\theta \cos\varphi)}{\partial r} \frac{\partial(r \sin\theta \cos\varphi)}{\partial r} + \frac{\partial(r \sin\theta \sin\varphi)}{\partial r} \frac{\partial(r \sin\theta \sin\varphi)}{\partial r} + \frac{\partial(r \cos\theta)}{\partial r} \frac{\partial(r \cos\theta)}{\partial r}$

$\quad = \sin^2\theta \cos^2\varphi + \sin^2\theta \sin^2\varphi + \cos^2\theta = \sin^2\theta \left(\cos^2\varphi + \sin^2\varphi\right) + \cos^2\theta = 1.$

$g'_{22} = \frac{\partial x^l}{\partial x'^2}\frac{\partial x_l}{\partial x'^2} = \frac{\partial(r\sin\theta\cos\varphi)}{\partial\theta}\frac{\partial(r\sin\theta\cos\varphi)}{\partial\theta} + \frac{\partial(r\sin\theta\sin\varphi)}{\partial\theta}\frac{\partial(r\sin\theta\sin\varphi)}{\partial\theta} + \frac{\partial(r\cos\theta)}{\partial\theta}\frac{\partial(r\cos\theta)}{\partial\theta}$
$\quad = r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta = r^2\cos^2\theta\left(\cos^2\varphi+\sin^2\varphi\right) + r^2\sin^2\theta = r^2.$
$g'_{33} = \frac{\partial x^l}{\partial x'^3}\frac{\partial x_l}{\partial x'^3} = \frac{\partial(r\sin\theta\cos\varphi)}{\partial\varphi}\frac{\partial(r\sin\theta\cos\varphi)}{\partial\varphi} + \frac{\partial(r\sin\theta\sin\varphi)}{\partial\varphi}\frac{\partial(r\sin\theta\sin\varphi)}{\partial\varphi} + \frac{\partial(r\cos\theta)}{\partial\varphi}\frac{\partial(r\cos\theta)}{\partial\varphi}$
$\quad = r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi + 0 = r^2\sin^2\theta.$
Nicht-diagonale Elemente:
Da der metrische Tensor symmetrisch ist, $g'_{ij} = g'_{ji}$, erhält man:
$g'_{12} = g'_{21} = \frac{\partial x^l}{\partial x'^1}\frac{\partial x_l}{\partial x'^2} = \frac{\partial(r\sin\theta\cos\varphi)}{\partial r}\frac{\partial(r\sin\theta\cos\varphi)}{\partial\theta} + \frac{\partial(r\sin\theta\sin\varphi)}{\partial r}\frac{\partial(r\sin\theta\sin\varphi)}{\partial\theta} + \frac{\partial(r\cos\theta)}{\partial r}\frac{\partial(r\cos\theta)}{\partial\theta}$
$\quad = \sin\theta\cos\varphi r\cos\theta\cos\varphi + \sin\theta\sin\varphi r\cos\theta\sin\varphi - \cos\theta r\sin\theta$
$\quad = r\sin\theta\cos\theta\left(\cos^2\varphi+\sin^2\varphi\right) - r\sin\theta\cos\theta = 0.$
$g'_{13} = g'_{31} = \frac{\partial x^l}{\partial x'^1}\frac{\partial x_l}{\partial x'^3} = \frac{\partial(r\sin\theta\cos\varphi)}{\partial r}\frac{\partial(r\sin\theta\cos\varphi)}{\partial\varphi} + \frac{\partial(r\sin\theta\sin\varphi)}{\partial r}\frac{\partial(r\sin\theta\sin\varphi)}{\partial\varphi} + \frac{\partial(r\cos\theta)}{\partial r}\frac{\partial(r\cos\theta)}{\partial\varphi}$
$\quad = -\sin\theta\cos\varphi r\sin\theta\sin\varphi + \sin\theta\sin\varphi r\sin\theta\cos\varphi + 0 = 0.$
$g'_{23} = g'_{32} = \frac{\partial x^l}{\partial x'^2}\frac{\partial x_l}{\partial x'^3} = \frac{\partial(r\sin\theta\cos\varphi)}{\partial\vartheta}\frac{\partial(r\sin\theta\cos\varphi)}{\partial\varphi} + \frac{\partial(r\sin\theta\sin\varphi)}{\partial\vartheta}\frac{\partial(r\sin\theta\sin\varphi)}{\partial\varphi} + \frac{\partial(r\cos\theta)}{\partial\vartheta}\frac{\partial(r\cos\theta)}{\partial\varphi}$
$\quad = -r\cos\theta\cos\varphi r\sin\theta\sin\varphi + r\cos\theta\sin\varphi\varphi r\sin\theta\cos\varphi + 0 = 0.$

$$\rightarrow g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

Metrik diagonal $\rightarrow$ Basis orthogonal.
b) Einheitsvektoren:
$\tilde{\mathbf{e}}'_i = \frac{\partial x^j}{\partial x'^i}\mathbf{e}_j$, also

$$\tilde{\mathbf{e}}'_r = \frac{\partial\mathbf{x}}{\partial r} = \frac{\partial}{\partial r}\begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \sin\theta\cos\varphi\mathbf{e}_x + \sin\theta\sin\varphi\mathbf{e}_y + \cos\theta\mathbf{e}_z.$$

$$\tilde{\mathbf{e}}'_\theta = \frac{\partial\mathbf{x}}{\partial\theta} = \frac{\partial}{\partial\theta}\begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix} = \begin{pmatrix} r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ -r\sin\theta \end{pmatrix}.$$

$$\tilde{\mathbf{e}}'_\varphi = \frac{\partial\mathbf{x}}{\partial\varphi} = \frac{\partial}{\partial\varphi}\begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix} = \begin{pmatrix} -r\sin\theta\sin\varphi \\ r\sin\theta\cos\varphi \\ 0 \end{pmatrix}.$$

Normieren:

$$\mathbf{e}'_r = \frac{1}{\sqrt{\sin^2\theta\cos^2\varphi+\sin^2\theta\sin^2\varphi+\cos^2\theta}}\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}.$$

$$\mathbf{e}'_\theta = \frac{1}{\sqrt{r^2\cos^2\theta\cos^2\varphi+r^2\cos^2\theta\sin^2\varphi+r^2\sin^2\theta}}\begin{pmatrix} r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ -r\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix}.$$

$$\mathbf{e}'_\varphi = \frac{1}{\sqrt{r^2\sin^2\theta\sin^2\varphi+r^2\sin^2\theta\cos^2\varphi}}\begin{pmatrix} -r\sin\theta\sin\varphi \\ r\sin\theta\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}.$$

c) Mit obiger Metrik: $U = 1$, $V = r$, $W = r\sin\theta$.
Gradient:
$\nabla\phi = \frac{1}{U}\mathbf{e}'^1\partial'_1\phi + \frac{1}{V}\mathbf{e}'^2\partial'_2\phi + \frac{1}{W}\mathbf{e}'^3\partial'_3\phi$
$\quad = \frac{1}{U}\mathbf{e}_r\partial_r\phi(r,\theta,\varphi) + \frac{1}{V}\mathbf{e}_\theta\partial_\theta\phi(r,\theta,\varphi) + \frac{1}{W}\mathbf{e}_\varphi\partial_\varphi\phi(r,\theta,\varphi)$
$\quad = \mathbf{e}_r\partial_r\phi(r,\theta,\varphi) + \frac{1}{r}\mathbf{e}_\theta\partial_\theta\phi(r,\theta,\varphi) + \frac{1}{r\sin\theta}\mathbf{e}_\varphi\partial_\varphi\phi(r,\theta,\varphi).$
Divergenz:
$\mathrm{div}\mathbf{a} = \frac{1}{UVW}\left[\partial'_1\left(VWa'^1\right) + \partial'_2\left(UWa'^2\right) + \partial'_3\left(UVa'^3\right)\right]$
$\quad = \frac{1}{UVW}\left[\partial_r\left(VWa_r\right) + \partial_\theta\left(UWa_\theta\right) + \partial_\varphi\left(UVa_\varphi\right)\right]$
$\quad = \frac{1}{r^2\sin\theta}\left[\partial_r\left(r^2\sin\theta a_r\right) + \partial_\theta\left(r\sin\theta a_\theta\right) + \partial_\varphi\left(ra_\varphi\right)\right]$
$\quad = \frac{1}{r^2}\partial_r\left(r^2a_r\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta a_\theta\right) + \frac{1}{r\sin\theta}\partial_\varphi a_\varphi.$