7. Tutorium - Lösungen **2.12.2011**

7.1 Delta-Folgen

a) Ja. Folge auf Testfunktion anwenden:
$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\varphi(x)dx = \lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{n}e^{-n\pi x^2}\varphi(x)dx = \left| \begin{array}{c} u = \sqrt{n\pi}x \\ du = \sqrt{n\pi}dx \end{array} \right|$
$= \lim_{n\to\infty}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\varphi(\frac{u}{\sqrt{n\pi}})du \to \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\varphi(0)du = \varphi(0)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}du = \varphi(0).$
b) Nein: $\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} f_\varepsilon(x)\varphi(x)dx = \lim_{\varepsilon\to 0}\int_{-\varepsilon^2/2}^{\varepsilon^2/2}\frac{1}{\varepsilon}\varphi(x)dx$
$\to \lim_{\varepsilon\to 0}\int_{-\varepsilon^2/2}^{\varepsilon^2/2}\frac{1}{\varepsilon}\varphi(0)dx = \varphi(0)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{2}\right) = \varphi(0)\times 0.$
c) Ja: $\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} f_\varepsilon(x)\varphi(x)dx = \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\left(\frac{1}{\varepsilon}-\frac{|x|}{\varepsilon^2}\right)\varphi(x)dx$
$\to \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\left(\frac{1}{\varepsilon}-\frac{|x|}{\varepsilon^2}\right)\varphi(0)dx = \varphi(0)\lim_{\varepsilon\to 0}\left[\int_{-\varepsilon}^{0}\left(\frac{1}{\varepsilon}+\frac{x}{\varepsilon^2}\right)dx + \int_0^{\varepsilon}\left(\frac{1}{\varepsilon}-\frac{x}{\varepsilon^2}\right)dx\right]$
$= \varphi(0)\lim_{\varepsilon\to 0}\left(\left.\frac{x}{\varepsilon}+\frac{x^2}{2\varepsilon^2}\right|_{x=-\varepsilon}^{0} + \left.\frac{x}{\varepsilon}-\frac{x^2}{2\varepsilon^2}\right|_{x=0}^{\varepsilon}\right) = \varphi(0)\left(0+1-\frac{1}{2}+1-\frac{1}{2}-0\right) = \varphi(0).$

7.2 Distributionen

a) $\delta(z^2-1) = \delta((z+1)(z-1)) = \frac{\delta(z+1)}{|2z|}+\frac{\delta(z-1)}{|2z|} = \frac{1}{2}\left[\delta(z+1)+\delta(z-1)\right]$. (Ableitung $f(x) = z^2-1$; $f'(x) = 2z$)
b) Integral über y: $y = -x$:
$I = \int_{-\infty}^{\infty} dx\delta(x^2+2x-24)f(x,-x) = \int_{-\infty}^{\infty} dx\delta\left((x-4)(x+6)\right)f(x,-x)$
Ableitung des Integranden: $2x+2$.
$I = \left.\frac{f(x,-x)}{|2x+2|}\right|_{x=4} + \left.\frac{f(x,-x)}{|2x+2|}\right|_{x=-6} = \frac{f(4,-4)}{10}+\frac{f(-6,6)}{10} = \frac{1}{10}\left[f(4,-4)+f(-6,6)\right].$
c) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(\sqrt{x_1^2+x_2^2+x_3^2}-R)\,d^3x = \int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\varphi\int_0^{\infty}r^2dr\delta(r-R) = 4\pi R^2.$
Die Lösung beschreibt den Flächeninhalt einer Kugelschale.

7.3 Verallgemeinerten Funktionen

a) $f(x) = \cos(x)\theta(\pi+x)\theta(\pi-x).$
$f'(x) = -\sin(x)\theta(\pi+x)\theta(\pi-x)+\cos(x)\delta(\pi+x)\theta(\pi-x)-\cos(x)\theta(\pi+x)\delta(\pi-x)$
$= -\sin(x)\theta(\pi+x)\theta(\pi-x)-\delta(\pi+x)+\delta(\pi-x).$
$f''(x) = -\cos(x)\theta(\pi+x)\theta(\pi-x)-0-0-\delta'(\pi+x)+\delta'(\pi-x)(-1)$
$= -\cos(x)\theta(\pi+x)\theta(\pi-x)-\delta'(x+\pi)+\delta'(x-\pi)$
$= -f(x)+\delta'(x-\pi)-\delta'(x+\pi).$
$f'''(x) = -f'(x)+\delta''(x-\pi)-\delta''(x+\pi).$
$n\geq 2: f^{(n)}(x) = -f^{(n-2)}+\delta^{(n-1)}(x-\pi)-\delta^{(n-1)}(x+\pi).$
b)$\left(\frac{d}{dx}-\omega\right)\left[\theta(x)e^{\omega x}+\theta(-x)e^{-\omega x}\right]$
$= \delta(x)e^{\omega x}+\theta(x)\omega e^{x}+\delta(-x)(-1)e^{-\omega x}+\theta(-x)(-\omega)e^{-\omega x}-\omega\theta(x)e^{\omega x}-\omega\theta(-x)e^{-\omega x}$
$= \delta(x)+\theta(x)\omega e^{x}-\delta(x)-\theta(-x)\omega e^{-\omega x}-\omega\theta(x)e^{\omega x}-\omega\theta(-x)e^{-\omega x}$
$= -2\theta(-x)\omega e^{-\omega x}$
$= -2\omega e^{-\omega x}\left[1-\theta(x)\right].$

7.4 Fourier Transformation

$k^2+1 = (k+i)(k-i)$. Pole liegen bei $k = \pm i$.
Berechnung mittels Residuensatz. $k = \mathrm{Re}k + i\mathrm{Im}k$. $e^{ikx} = e^{ix\mathrm{Re}k}e^{-x\mathrm{Im}k}$.
Exponentielle Dämpfung für $x\mathrm{Im}k > 0$, also

$x>0$, Im$k>0$: Schließen eines harmlosen Großkreises in der oberen Halbebene.
$x<0$, Im$k<0$: Schließen eines harmlosen Großkreises in der unteren Halbebene.
Daher:

$$I(x) = 2\pi i \mathrm{Res}_{k=i}\left[\frac{1}{2\pi}\frac{e^{ikx}}{(k+i)(k-i)}\right]\theta(x) - 2\pi i \mathrm{Res}_{k=-i}\left[\frac{1}{2\pi}\frac{e^{ikx}}{(k+i)(k-i)}\right]\theta(-x)$$

(negatives Vorzeichen vor zweitem Term wegen negativem Umlaufsinn durch Schließen in unterer Halbebene).

$$= i\frac{e^{-x}}{2i}\theta(x) - i\frac{e^{x}}{-2i}\theta(-x) = \frac{e^{-x}}{2}\theta(x) + \frac{e^{x}}{2}\underbrace{\theta(-x)}_{1-\theta(x)}$$

$$= \frac{e^{x}}{2} + \theta(x)\left(\frac{e^{-x}}{2} - \frac{e^{x}}{2}\right) = \frac{e^{x}}{2} - \theta(x)\sinh(x).$$

7.5 Indexschreibweise

a) $\mathrm{rot}(\mathbf{x}\times\mathbf{a})\rightarrow\varepsilon_{ijk}\nabla_j\varepsilon_{klm}x_l a_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\underbrace{\nabla_j x_l}_{\delta_{jl}} a_m = a_i - \underbrace{\delta_{jj}}_{3} a_i = -2a_i.$

b) $\mathrm{rot}\left(\mathbf{x}\theta(\mathbf{a}\cdot\mathbf{x})\right)\rightarrow\varepsilon_{ijk}\nabla_j x_k\theta(a_l x_l) = \varepsilon_{ijk}\left[\underbrace{(\nabla_j x_k)}_{\delta_{jk}}\theta(a_l x_l) + x_k \underbrace{\nabla_j\theta(a_l x_l)}_{\delta(a_l x_l)\nabla_j(a_m x_m)}\right]$
$= \underbrace{\varepsilon_{ijk}\delta_{jk}}_{=0}\theta(a_l x_l) + \varepsilon_{ijk}x_k\delta(a_l x_l)a_m\delta_{jm} = \varepsilon_{ijk}a_j x_k\delta(a_l x_l)\rightarrow (\mathbf{a}\times\mathbf{x})\,\delta(\mathbf{a}\cdot\mathbf{x}).$

7.6 Maxwell-Gleichungen

a) $\nabla_i E_i = 4\pi\rho,$
$\nabla_i B_i = 0,$
$\varepsilon_{klm}\nabla_l B_m = 4\pi j_k + \frac{\partial}{\partial t}E_k,$
$\varepsilon_{klm}\nabla_l E_m = -\frac{\partial}{\partial t}B_k,$
$f_k = \rho E_k + \varepsilon_{klm}j_l B_m.$
b) $\mathrm{div}\mathbf{j} = \mathrm{div}\left(\frac{1}{4\pi}\mathrm{rot}\mathbf{B} - \frac{1}{4\pi}\frac{\partial}{\partial t}\mathbf{E}\right)$
$\rightarrow\nabla_k j_k = \nabla_k\left(\frac{1}{4\pi}\varepsilon_{klm}\nabla_l B_m - \frac{1}{4\pi}\frac{\partial}{\partial t}E_k\right)$
$= \frac{1}{4\pi}\underbrace{\varepsilon_{klm}\nabla_k\nabla_l}_{=0} B_m - \frac{1}{4\pi}\frac{\partial}{\partial t}\underbrace{\nabla_k E_k}_{4\pi\rho}$
$= -\frac{\partial}{\partial t}\rho.$
Also, Kontinuitätsgleichung: $\mathrm{div}\mathbf{j} + \frac{\partial}{\partial t}\rho = 0.$
c) $\mathrm{div}\mathbf{B} = 0 \rightarrow \nabla_i B_i = 0.$
$\rightarrow\nabla_i\left(\varepsilon_{ikm}\nabla_k A_m\right) = \underbrace{\varepsilon_{ikm}\nabla_i\nabla_k}_{=0} A_m = 0.$ Ok.
$\mathrm{rot}\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}\rightarrow\varepsilon_{klm}\nabla_l E_m = -\frac{\partial}{\partial t}B_k$
$\rightarrow\varepsilon_{klm}\nabla_l\left(-\nabla_m\phi - \frac{\partial}{\partial t}A_m\right) = -\frac{\partial}{\partial t}\left(\varepsilon_{klm}\nabla_l A_m\right)$
$\rightarrow-\underbrace{\varepsilon_{klm}\nabla_l\nabla_m}_{=0}\phi - \varepsilon_{klm}\frac{\partial}{\partial t}\nabla_l A_m = -\varepsilon_{klm}\frac{\partial}{\partial t}\nabla_l A_m.$ Ok.