8. Tutorium - Lösungen **16.12.2011**

8.1 Greensche Funktion 1

a) $\mathcal{L}_t = \frac{d}{dt} + 1$, $f(t) = e^t$, $\mathcal{L}_t y(t) = f(t)$.
Ansatz: $G(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k) e^{ik(t-t')} dk$
und $\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(t-t')} dk$ einsetzen in
$\mathcal{L}_t G(t,t') = \delta(t-t')$,
$\left(\frac{d}{dt}+1\right) G(t,t') = \delta(t-t')$,
$\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k)\left(\frac{d}{dt}+1\right) e^{ik(t-t')} dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(t-t')} dk$,
$\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k)\left(ik+1\right) e^{ik(t-t')} dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(t-t')} dk$.
Vergleich der Integranden:
$\tilde{G}(k)(ik+1) = 1. \quad \rightarrow \quad \tilde{G}(k) = \frac{1}{ik+1}$.
$G(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik(t-t')}}{ik+1} dk$.
Pol liegt bei $k = +i$ im oberen Bereich. Für $t - t' > 0$: Großkreis oben schließen ($ik = i(\mathrm{Re}k + i\mathrm{Im}k) = i\mathrm{Re}k - \mathrm{Im}k$: Für $\mathrm{Im}k > 0$ exponentiell gedämpft). Für $t - t' < 0$: Großkreis unten schließen: Kein Pol eingeschlossen.
$G(t,t') = \theta(t-t') 2\pi i \mathrm{Res}_{k\to i} \frac{1}{2\pi}\frac{e^{ik(t-t')}}{ik+1} = \theta(t-t')2\pi i \lim_{k\to i}(k-i)\frac{1}{2\pi}\frac{e^{ik(t-t')}}{i(k-i)} = \theta(t-t')e^{-(t-t')}$.
b) Randbedingung: $y(0) = 0 \quad \rightarrow \quad G(0,t') = \theta(-t')e^{t'} = 0$ für $t' > 0$. Ok.
Lösung: $y(t) = \int_0^{\infty} G(t,t')f(t')dt' = \int_0^{\infty}\theta(t-t')e^{-(t-t')}e^{t'}dt' = \int_0^t e^{-t}e^{2t'}dt' = e^{-t}\frac{e^{2t}-1}{2} = \frac{e^t - e^{-t}}{2} = \sinh t$.
c) <u>Probe</u>: $\left(\frac{d}{dt}+1\right)\sinh t = \cosh t + \sinh t = \frac{e^t+e^{-t}}{2} + \frac{e^t - e^{-t}}{2} = e^t$.
$y(0) = \sinh 0 = 0$.

8.2 Greensche Funktion 2

a) $\mathcal{L}_x = -\frac{d^2}{dx^2} - \omega\frac{d}{dx} + 2\omega^2$, $f(x) = e^{\omega x}$, $\mathcal{L}_x y(x) = f(x)$.
Ansatz: $G(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(k)e^{ik(x-x')}dk$
und $\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}dk$ einsetzen in
$\mathcal{L}_x G(x,x') = \delta(x-x')$,
$\left(-\frac{d^2}{dx^2} - \omega\frac{d}{dx} + 2\omega^2\right) G(x,x') = \delta(x-x')$,
$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(k)\left(-\frac{d^2}{dx^2} - \omega\frac{d}{dx} + 2\omega^2\right) e^{ik(x-x')}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}dk$,
$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(k)\left(k^2 - i\omega k + 2\omega^2\right) e^{ik(x-x')}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}dk$.
Vergleich der Integranden:
$\tilde{G}(k)\left(k^2 - i\omega k + 2\omega^2\right) = 1. \quad \rightarrow \quad \tilde{G}(k) = \frac{1}{k^2 - i\omega k + 2\omega^2} = \frac{1}{(k-2i\omega)(k+i\omega)}$.
$G(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik(x-x')}}{(k-2i\omega)(k+i\omega)}dk$.
Ein Pol liegt bei $k = +2i\omega$ im oberen Bereich, ein Pol bei $k = -i\omega$ im unteren Bereich. Für $x - x' > 0$: Großkreis oben schließen ($ik = i(\mathrm{Re}k + i\mathrm{Im}k) = i\mathrm{Re}k - \mathrm{Im}k$: Für $\mathrm{Im}k > 0$ exponentiell gedämpft). Für $x - x' < 0$: Großkreis unten schließen (Vorzeichen von Umlaufsinn).
$G(x,x') = \theta(x-x')2\pi i\mathrm{Res}_{k\to 2i\omega}\frac{1}{2\pi}\frac{e^{ik(x-x')}}{(k-2i\omega)(k+i\omega)} + \theta(x'-x)2\pi i\mathrm{Res}_{k\to -i\omega}\frac{1}{2\pi}\frac{e^{ik(x-x')}}{(k-2i\omega)(k+i\omega)}(-1)$
$= \theta(x-x')2\pi i\lim_{k\to 2i\omega}(k-2i\omega)\frac{1}{2\pi}\frac{e^{ik(x-x')}}{(k-2i\omega)(k+i\omega)} - \theta(x'-x)2\pi i\lim_{k\to -i\omega}(k+i\omega)\frac{1}{2\pi}\frac{e^{ik(x-x')}}{(k-2i\omega)(k+i\omega)}$
$= \theta(x-x')i\frac{e^{-2\omega(x-x')}}{3i\omega} - \theta(x'-x)i\frac{e^{\omega(x-x')}}{-3i\omega} = \theta(x-x')\frac{e^{-2\omega(x-x')}}{3\omega} + \theta(x'-x)\frac{e^{\omega(x-x')}}{3\omega}$.
b) Randbedingung: $G(0, x' > 0) = \frac{1}{3\omega}\left[0 + e^{-x'\omega}\right]$ nicht erfüllt.
Homogene Greensche Funktion über Ansatz:
$G = G_I + Ae^{-2\omega(x-x')} + Be^{\omega(x-x')}$.

$G(0,x') = \frac{e^{-x'\omega}}{3\omega} + Ae^{2\omega x'} + Be^{-\omega x'} = 0.$ (I)
$G'(x,x') = \frac{1}{3\omega}\left[\delta(x-x')e^{-2\omega(x-x')} + (-2\omega)\theta(x-x')e^{-2\omega(x-x')} + (-1)\delta(x'-x)e^{\omega(x-x')} + \omega\theta(x'-x)e^{\omega(x-x')}\right]$
$+A(-2\omega)e^{-2\omega(x-x')} + B\omega e^{\omega(x-x')}.$
$G'(0,x') = -\frac{2}{3}\theta(-x')... + \frac{1}{3}e^{-\omega x'} - A2\omega e^{2\omega x'} + B\omega e^{-\omega x'} = 0.$ (II)
Aus (I) und (II) folgt: $A = 0$, $B = -\frac{1}{3\omega}$.
$\rightarrow G(x,x') = G_I - \frac{1}{3\omega}e^{\omega(x-x')}$.
Lösung: $y(x) = \int_0^\infty G(x,x')f(x')dx' = \int_0^x dx' \frac{e^{-2\omega(x-x')}e^{\omega x'}}{3\omega} + \int_x^\infty dx' \frac{e^{\omega(x-x')}}{3\omega}e^{\omega x'} - \int_0^\infty dx' \frac{e^{\omega(x-x')}}{3\omega}e^{\omega x'}$
$= \frac{1}{3\omega}\left(e^{-2\omega x}\int_0^x dx' e^{+3\omega x'} - e^{\omega x}x\right)$
$= \frac{1}{3\omega}\left(e^{-2\omega x}\frac{e^{3\omega x}-1}{3\omega}\right) - \frac{xe^{\omega x}}{3\omega}$
$= \frac{1}{9\omega^2}\left(e^{\omega x} - e^{-2\omega x}\right) - \frac{xe^{\omega x}}{3\omega}.$
<u>Probe</u>: $y(0) = 0$,
$y'(0) = \frac{1}{9\omega^2}(\omega + 2\omega) - \frac{1}{3\omega} - 0 = 0.$
$\frac{d}{dx}y(x) = \frac{1}{9\omega^2}\left(e^{\omega x}\omega + 2\omega e^{-2\omega x}\right) - \frac{e^{\omega x} + xe^{\omega x}\omega}{3\omega} = -\frac{2}{9}\frac{e^{\omega x}}{\omega} + \frac{2}{9}\frac{e^{-2\omega x}}{\omega} - \frac{x}{3}e^{\omega x}.$
$\frac{d^2}{dx^2}y(x) = -\frac{2}{9}e^{\omega x} - \frac{4}{9}e^{-2\omega x} - \frac{e^{\omega x}}{3} - \frac{x\omega}{3}e^{\omega x} = -\frac{5}{9}e^{\omega x} - \frac{4}{9}e^{-2\omega x} - \frac{x\omega}{3}e^{\omega x}.$
$\mathcal{L}_x y(x) = -\frac{d^2}{dx^2}y(x) - \omega\frac{d}{dx}y(x) + 2\omega^2 y(x)$
$= \frac{5}{9}e^{\omega x} + \frac{4}{9}e^{-2\omega x} + \frac{x\omega}{3}e^{\omega x} + \frac{2}{9}e^{\omega x} - \frac{2}{9}e^{-2\omega x} + \frac{x\omega}{3}e^{\omega x} + \frac{2}{9}e^{\omega x} - \frac{2}{9}e^{-2\omega x} - \frac{2x\omega}{3}e^{\omega x}$
$= \frac{9}{9}e^{\omega x} + 0e^{-2\omega x} + 0e^{\omega x} = e^{\omega x} = f(x).$

8.3 Sturm-Liouville-Problem

a)
$\mathcal{L}_x y(x) = -x^2y'' + xy' + y \stackrel{!}{=} a_0(x)y(x) + a_1(x)\frac{d}{dx}y(x) + a_2(x)\frac{d^2}{dx^2}y(x).$
$a_0(x) = 1$, $a_1(x) = x$, $a_2(x) = -x^2$.
$\rightarrow$Transformation auf Sturm-Liouville Form $\mathcal{S}_x y(x) = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right]y(x) + q(x)y(x)$
mit $p(x) = e^{\int\frac{a_1(x)}{a_2(x)}dx} = \exp\left(\int\frac{x}{-x^2}dx\right) = \exp\left(-\int\frac{1}{x}dx\right) = \exp\left(-\ln x + c\right) = \frac{1}{x}\exp c = \frac{\tilde{c}}{x}.$
$q(x) = p(x)\frac{a_0(x)}{a_2(x)} = \frac{\tilde{c}}{x(-x^2)} = -\frac{\tilde{c}}{x^3}.$
$\mathcal{S}_x y(x) = \frac{d}{dx}\left(\frac{\tilde{c}}{x}\frac{d}{dx}\right)y(x) - \frac{\tilde{c}}{x^3}y(x)$
mit $\tilde{c} = 1$: $\mathcal{L}_x y(x) = \frac{a_2(x)}{p(x)}\mathcal{S}_x y(x) = -x^3\left[\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\right)y(x) - \frac{1}{x^3}y(x)\right] = -x^3\left[\left(\frac{y'}{x}\right)' - \frac{y}{x^3}\right].$
<u>Probe</u>: $-x^3\left[\left(\frac{y'}{x}\right)' - \frac{y}{x^3}\right] = -x^3\left[\frac{y''}{x} - \frac{y'}{x^2} - \frac{y}{x^3}\right] = -x^2y'' + xy' + y = \mathcal{L}_x y(x).$
b) $\mathcal{L}_x y(x) = f(x) = \lambda y \quad \rightarrow \quad \mathcal{S}_x y(x) = F(x),$
$F(x) = p(x)\frac{f(x)}{a_2(x)} = \frac{\tilde{c}\lambda y}{x(-x^2)} = -\frac{\tilde{c}\lambda}{x^3}y.$
$\left[\mathcal{S}_x + \lambda\rho(x)\right]y(x) = 0 \quad \leftrightarrow \quad \left[\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\right) - \frac{1}{x^3} + \frac{\lambda}{x^3}\right]y(x) = 0$ für $x \in [a,b]$.
also $\rho(x) = \frac{1}{x^3}$, $p(x) = \frac{1}{x}$
Sturm-Liouville Transformation:
$\xi = t(x) = \int_a^x\sqrt{\frac{\rho(s)}{p(s)}}ds = \int_a^x\sqrt{\frac{s}{s^3}}ds = \int_a^x\frac{1}{s}ds = \ln x - \ln a = \ln\frac{x}{a} \quad \rightarrow \quad x(t) = ae^t.$
$w(t) = \sqrt[4]{p(x(t))\rho(x(t))}y(x(t)) = \sqrt[4]{\frac{1}{x(t)}\frac{1}{x^3(t)}}y(x(t)) = \frac{y(x(t))}{x(t)}.$
$\hat{q}(t) = \frac{1}{\rho}\left[-q - \sqrt[4]{p\rho}\left(p\left(\frac{1}{\sqrt[4]{p\rho}}\right)'\right)'\right] = x^3\left[\frac{1}{x^3} - \sqrt[4]{\frac{1}{x}\frac{1}{x^3}}\left(\frac{1}{x}\left(\frac{1}{\sqrt[4]{\frac{1}{x}\frac{1}{x^3}}}\right)'\right)'\right]$
$= x^3\left[\frac{1}{x^3} - \frac{1}{x}\left(\frac{1}{x}(x)'\right)'\right] = x^3\left[\frac{1}{x^3} - \frac{1}{x}\left(\frac{1}{x}\right)'\right] = x^3\left[\frac{1}{x^3} - \frac{1}{x}\left(-\frac{1}{x^2}\right)\right] = 2.$
$c = t(b) = \ln\frac{b}{a}.$
Liouvillesche Normalform:
$-\frac{d^2}{dt^2}w(t) + \left[\hat{q}(t) - \lambda\right]w(t) = 0$ für $t \in [0,c]$,
$-\frac{d^2}{dt^2}w(t) + \left[2 - \lambda\right]w(t) = 0$ für $t \in [0, \ln\frac{b}{a}]$.
<u>Probe</u>: $w(t) = \frac{y(x(t))}{x(t)} = \frac{y(ae^t)}{ae^t}.$

$\frac{d^2}{dt^2} w(t) = \frac{d^2}{dt^2} \left(\frac{y(ae^t)}{ae^t} \right) = \frac{d}{dt} \left(\frac{y'(ae^t)ae^t}{ae^t} - \frac{y(ae^t)}{(ae^t)^2} ae^t \right) = \frac{d}{dt} \left(y'(ae^t) - \frac{y(ae^t)}{ae^t} \right)$
$= y''(ae^t)ae^t - \frac{y'(ae^t)ae^t}{ae^t} + \frac{y(ae^t)}{(ae^t)^2} ae^t = y''(x)x - y'(x) + \frac{y(x)}{x}$
$\rightarrow -\frac{d^2}{dt^2} w(t) + [2 - \lambda]\, w(t) = -y''(x)x + y'(x) - \frac{y(x)}{x} + 2\frac{y(x)}{x} - \lambda \frac{y(x)}{x} = 0.$
$\overset{\times x}{\rightarrow} -x^2 y'' + xy' - y + 2y - \lambda y = 0 = \mathcal{L}_x y - \lambda y.$