9. Tutorium - Lösungen **13.01.2012**

9.1 Separationsansatz 1

$\Box\Phi \equiv \left(\Delta - \frac{\partial^2}{\partial t^2}\right)\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right)\Phi = \lambda\Phi.$
Ansatz: $\Phi(x,y,z,t) = \Phi_1(x)\Phi_2(y)\Phi_3(z)\Phi_4(t)$.
$\left[\frac{\partial^2}{\partial x^2}\Phi_1(x)\right]\Phi_2(y)\Phi_3(z)\Phi_4(t) + \Phi_1(x)\left[\frac{\partial^2}{\partial y^2}\Phi_2(y)\right]\Phi_3(z)\Phi_4(t) + \Phi_1(x)\Phi_2(y)\left[\frac{\partial^2}{\partial z^2}\Phi_3(z)\right]\Phi_4(t)$
$-\Phi_1(x)\Phi_2(y)\Phi_3(z)\left[\frac{\partial^2}{\partial t^2}\Phi_4(t)\right] = \lambda\Phi_1(x)\Phi_2(y)\Phi_3(z)\Phi_4(t).$
Ganze Gleichung durch $\Phi = \Phi_1\Phi_2\Phi_3\Phi_4$ dividieren:
$\frac{1}{\Phi_1(x)}\left[\frac{\partial^2}{\partial x^2}\Phi_1(x)\right] + \frac{1}{\Phi_2(y)}\left[\frac{\partial^2}{\partial y^2}\Phi_2(y)\right] + \frac{1}{\Phi_3(z)}\left[\frac{\partial^2}{\partial z^2}\Phi_3(z)\right] - \frac{1}{\Phi_4(t)}\left[\frac{\partial^2}{\partial t^2}\Phi_4(t)\right] = \lambda.$
Nun der Reihe nach die Terme abspalten:
$\frac{1}{\Phi_1(x)}\left[\frac{\partial^2}{\partial x^2}\Phi_1(x)\right] + \frac{1}{\Phi_2(y)}\left[\frac{\partial^2}{\partial y^2}\Phi_2(y)\right] + \frac{1}{\Phi_3(z)}\left[\frac{\partial^2}{\partial z^2}\Phi_3(z)\right] = \frac{1}{\Phi_4(t)}\left[\frac{\partial^2}{\partial t^2}\Phi_4(t)\right] + \lambda = A(x,y,z) = A(t) = A = const.$
$\rightarrow \Phi_4''(t) = (A-\lambda)\,\Phi_4(t).$
$\frac{1}{\Phi_1}\Phi_1'' + \frac{1}{\Phi_2}\Phi_2'' = -\frac{1}{\Phi_3}\Phi_3'' + A = B(x,y) = B(z) = B = const.$
$\rightarrow \Phi_3''(z) = (A-B)\,\Phi_3(z).$
$\frac{1}{\Phi_1}\Phi_1'' = -\frac{1}{\Phi_2}\Phi_2'' + B = C(x) = C(y) = C = const.$
$\rightarrow \Phi_2''(y) = (B-C)\,\Phi_2(y).$
$\rightarrow \Phi_1''(x) = C\Phi_1(x).$
Alternativer Lösungsweg:

$$\underbrace{\frac{\Phi_1''(x)}{\Phi_1(x)}}_{\alpha(x)=\alpha=const} + \underbrace{\frac{\Phi_2''(y)}{\Phi_2(y)}}_{\beta(y)=\beta=const} + \underbrace{\frac{\Phi_3''(z)}{\Phi_3(z)}}_{\gamma(z)=\gamma=const} - \underbrace{\frac{\Phi_4''(t)}{\Phi_4(t)}}_{\delta(t)=\delta=const} = \lambda = \Lambda(\alpha,\beta,\gamma,\delta) = const.$$

Da die rechte Seite eine Konstante ist, muss auch die linke Seite konstant sein, und da jeder der vier Terme von einer anderen Variable abhängt, muss jeder der vier Terme konstant sein.
$\rightarrow \Phi_1''(x) = \alpha\Phi_1(x)$,
$\Phi_2''(y) = \beta\Phi_2(y)$,
$\Phi_3''(z) = \gamma\Phi_3(z)$,
$\Phi_4''(t) = \delta\Phi_4(t)$,
mit der Zusatzbedingung $\alpha + \beta + \gamma - \delta = \lambda$.
Das entspricht der ersten Lösung mit $\alpha = C$, $\beta = B - C$, $\gamma = A - B$, $\delta = A - \lambda$.

9.2 Separationsansatz 2

Zylinderkoordinaten: $\mathbf{x} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$.
$\bar{x}_1 = r$, $\bar{x}_2 = \varphi$, $\bar{x}_3 = z$.
$U^2 = \frac{\partial x_i}{\partial \bar{x}_1}\frac{\partial x_i}{\partial \bar{x}_1} = \cos^2\varphi + \sin^2\varphi + 0 = 1.$
$V^2 = \frac{\partial x_i}{\partial \bar{x}_2}\frac{\partial x_i}{\partial \bar{x}_2} = r^2\left(\sin^2\varphi + \cos^2\varphi\right) + 0 = r^2.$
$W^2 = \frac{\partial x_i}{\partial \bar{x}_2}\frac{\partial x_i}{\partial \bar{x}_2} = 0 + 0 + 1 = 1.$
Laplace-Operator: $\Delta = \frac{1}{UVW}\left[\frac{\partial}{\partial \bar{x}_1}\left(\frac{VW}{U}\frac{\partial}{\partial \bar{x}_1}\right) + \text{zykl. Perm.}\right]$
$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \varphi}\frac{1}{r}\frac{\partial}{\partial \varphi} + \frac{\partial}{\partial z}r\frac{\partial}{\partial z}\right]$
$= \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$
Homogene Laplace-Gleichung: $\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right)\Phi(r,\varphi,z) = 0.$

Ansatz: $\Phi(r,\varphi,z)=\Phi_1(r)\Phi_2(\varphi)\Phi_3(z)$.
$\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}\right)\Phi_1(r)\right]\Phi_2(\varphi)\Phi_3(z)+\frac{1}{r^2}\Phi_1(r)\left[\frac{\partial^2}{\partial\varphi^2}\Phi_2(\varphi)\right]\Phi_3(z)+\Phi_1(r)\Phi_2(\varphi)\left[\frac{\partial^2}{\partial z^2}\Phi_3(z)\right]=0.$
$\left[\frac{1}{\Phi_1(r)}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}\right)\Phi_1(r)\right]+\frac{1}{r^2}\frac{1}{\Phi_2(\varphi)}\left[\frac{\partial^2}{\partial\varphi^2}\Phi_2(\varphi)\right]=-\frac{1}{\Phi_3(z)}\left[\frac{\partial^2}{\partial z^2}\Phi_3(z)\right]=A(r,\varphi)=A(z)=A=const.$
$\rightarrow\Phi_3''(z)=-A\Phi_3(z)$.
$r^2\left[\frac{1}{\Phi_1(r)}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}\right)\Phi_1(r)-A\right]=-\frac{1}{\Phi_2(\varphi)}\left[\frac{\partial^2}{\partial\varphi^2}\Phi_2(\varphi)\right]=B(r)=B(\varphi)=B=const.$
$\rightarrow\Phi_2''(\varphi)=-B\Phi_2(\varphi)$.
$\rightarrow\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}\right)\Phi_1(r)=A\Phi_1(r)+\frac{B}{r^2}\Phi_1(r)$.

9.3 Fuchssche Klasse

a) $zw''+4w'+2\frac{w}{z}=0. \quad\rightarrow\quad w''+\frac{4}{z}w'+\frac{2}{z^2}w=0=w''+p_1(z)w'+p_2(z)w.$
$\rightarrow p_1(z)=\frac{4}{z}$, $p_2(z)=\frac{2}{z^2}$.
Singuläre Punkte: $z_0=0$:
$\alpha_0=\lim_{z\to z_0}(z-z_0)p_1(z)=\lim_{z\to0}(z-0)\frac{4}{z}=4.$
$\beta_0=\lim_{z\to0}(z-z_0)p_2(z)=\lim_{z\to0}(z-0)^2\frac{2}{z^2}=2.$
Charakteristische Exponenten:
$\sigma^2+\sigma(\alpha_0-1)+\beta_0=0=\sigma^2+\sigma(4-1)+2=\sigma^2+3\sigma+2$
$=(\sigma+1)(\sigma+2)=0.$
$\rightarrow\sigma_1=-1$, $\sigma_2=-2$.
Singuläre Punkte um $z=\infty$: Transformation $t=1/z$:
$u''+\tilde{p}_1(t)u'+\tilde{p}_2(t)u=0$ mit $\tilde{p}_1=\frac{2}{t}-\frac{1}{t^2}p_1(\frac{1}{t})$, $\tilde{p}_2=\frac{1}{t^4}p_2(\frac{1}{t})$.
$\rightarrow\tilde{p}_1=\frac{2}{t}-\frac{1}{t^2}4t=-\frac{2}{t}$, $\tilde{p}_2=\frac{1}{t^4}2t^2=\frac{2}{t^2}$.
$\rightarrow u''-\frac{2}{t}u'+\frac{2}{t^2}u=0.$
Singuläre Punkte: $\alpha_0=-2$, $\beta_0=2$.
$\sigma^2+\sigma(\alpha_0-1)+\beta_0=0=\sigma^2+\sigma(-3)+\beta_0 2=(\sigma-1)(\sigma-2).$
$\rightarrow\sigma_1=1$, $\sigma_2=2$.
b) Im allgemeinen Fall würde man hier nur eine linear unabhängige Lösung erwarten (bei diesem speziellen Beispiel erhält man aber gleich beide):
Ansatz für generalisierte Potenzreihe: $w(z)=\sum_{n=0}^{\infty}w_n z^{n+\sigma}$.
$\rightarrow w'(z)=\sum_{n=0}^{\infty}w_n(n+\sigma)z^{n+\sigma-1}$,
$w''(z)=\sum_{n=0}^{\infty}w_n(n+\sigma)(n+\sigma-1)z^{n+\sigma-2}$.
Einsetzen liefert:
$zw''+4w'+2\frac{w}{z}=0=\sum_{n=0}^{\infty}z^{n+\sigma-1}w_n\left[(n+\sigma)(n+\sigma-1)+4(n+\sigma)+2\right].$
$\rightarrow w_n(n+\sigma+1)(n+\sigma+2)=0.$
Da $n\in\mathbb{N}_0$ ist das nur erfüllt 0 falls entweder $w_n=0$ oder $n=-1-\sigma$ oder $n=-2-\sigma$.
Mit $\sigma_1=-1$: $w_n n(n+1)=0\rightarrow w_0=C\neq0$, $w_{n>0}=0$.
Mit $\sigma_2=-2$: $w_n(n-1)n=0\rightarrow w_0=A\neq0$, $w_1=B\neq0$, $w_{n>1}=0$.
$\rightarrow w_1(z)=\frac{C}{z}$, $w_2(z)=\frac{A}{z^2}+\frac{B}{z}$.
Lösung: $w(z)=\frac{A}{z^2}+\frac{B}{z}$.
c) Einsetzen: $w'(z)=-2\frac{A}{z^3}-\frac{B}{z^2}$, $w''(z)=6\frac{A}{z^4}+2\frac{B}{z^3}$.
$zw''+4w'+2\frac{w}{z}=0=z\left(6\frac{A}{z^4}+2\frac{B}{z^3}\right)+4\left(-2\frac{A}{z^3}-\frac{B}{z^2}\right)+\frac{2}{z}\left(\frac{A}{z^2}+\frac{B}{z}\right)=0\frac{A}{z^3}+0\frac{B}{z^2}=0.$

9.4 Hypergeometrische Funktion

a) $F(1,\beta,\beta;z)=\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\frac{z^j}{j!}=\sum_{j=0}^{\infty}\frac{(1)_j(\beta)_j}{(\beta)_j}\frac{z^j}{j!}=\sum_{j=0}^{\infty}\frac{j!}{0!}\frac{z^j}{j!}=\sum_{j=0}^{\infty}z^j=\frac{1}{1-z}$,
mit dem Pochhammer Symbol $(a)_j=a\cdot(a+1)\cdot\ldots\cdot(a+j-1)=\Gamma(a+j)/\Gamma(a)=(a+j-1)!/(a-1)!$.
$F(\frac{3}{2},1,\frac{3}{2};z)=\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\frac{z^j}{j!}=\sum_{j=0}^{\infty}\frac{(\frac{3}{2})_j(1)_j}{(\frac{3}{2})_j}\frac{z^j}{j!}=\sum_{j=0}^{\infty}\frac{j!}{0!}\frac{z^j}{j!}=\sum_{j=0}^{\infty}z^j=\frac{1}{1-z}$,
b) $F(1,\frac{3}{2},\frac{5}{2};x)=\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\frac{x^j}{j!}=\sum_{j=0}^{\infty}\frac{(1)_j(\frac{3}{2})_j}{(\frac{5}{2})_j}\frac{x^j}{j!}=\sum_{j=0}^{\infty}\frac{3}{2j+3}x^j$
mit $\frac{(\frac{3}{2})_j}{(\frac{5}{2})_j}=\frac{3\cdot5\cdot\ldots\cdot(2j-1)\cdot(2j+1)}{5\cdot7\cdot\ldots\cdot(2j+1)\cdot(2j+3)}=\frac{3}{2j+3}$.
$F(1,\frac{3}{2},\frac{5}{2};x)=\sum_{j=0}^{\infty}\frac{3}{2j+3}x^j=3\sum_{j=0}^{\infty}\frac{1}{2j+3}x^j=3\sum_{j=0}^{\infty}\frac{1}{2j+3}\left(\sqrt{x}\right)^{2j}=3\sum_{j=1}^{\infty}\frac{1}{2j+1}\left(\sqrt{x}\right)^{2j-2}$

$= \frac{3}{(\sqrt{x})^3} \sum_{j=1}^{\infty} \frac{1}{2j+1} (\sqrt{x})^{2j+1} = \frac{3}{(\sqrt{x})^3} \left[\sum_{j=0}^{\infty} \frac{1}{2j+1} (\sqrt{x})^{2j+1} - \frac{1}{1} (\sqrt{x})^1 \right]$
$= \frac{3}{(\sqrt{x})^3} \left[\frac{1}{2} \times 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} (\sqrt{x})^{2k-1} - \sqrt{x} \right] = \frac{3}{(\sqrt{x})^3} \left[\frac{1}{2} \times \log \left(\frac{1+\sqrt{x}}{1-\sqrt{x}} \right) - \sqrt{x} \right]$
$= 3 \left[\frac{1}{2x^{3/2}} \log \left(\frac{1+x^{1/2}}{1-x^{1/2}} \right) - \frac{1}{x} \right].$

9.5 Legendre Polynome

$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2-2\mathbf{r}\cdot\mathbf{r}'+r'^2}} = \frac{1}{r} \frac{1}{\sqrt{1-2\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\left(\frac{r'}{r}\right)+\left(\frac{r'}{r}\right)^2}} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r} \right)^l P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}').$
Mit $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$, $P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$:
$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} \left[\left(\frac{r'}{r} \right)^0 P_0(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') + \left(\frac{r'}{r} \right)^1 P_1(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') + \left(\frac{r'}{r} \right)^2 P_2(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') + \left(\frac{r'}{r} \right)^3 P_3(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') + O\left(\frac{r'^4}{r^4} \right) \right]$
$= \frac{1}{r} \left[1 + \frac{r'}{r} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' + \left(\frac{r'}{r} \right)^2 \frac{1}{2} \left(3 \left(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' \right)^2 - 1 \right) + \left(\frac{r'}{r} \right)^3 \frac{1}{2} \left(5 \left(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' \right)^3 - 3\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' \right) + O\left(\frac{r'^4}{r^4} \right) \right]$
$= \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \frac{3\left(\mathbf{r}\cdot\mathbf{r}'\right)^2 - r^2 r'^2}{2r^5} + \frac{5\left(\mathbf{r}\cdot\mathbf{r}'\right)^3 - 3\left(\mathbf{r}\cdot\mathbf{r}'\right) r^2 r'^2}{2r^7} + O\left(\frac{r'^4}{r^5} \right)$