1. Test - Lösungen **7.12.2012**

1 Indexschreibweise

$\mathrm{rot}\left(\left(\mathbf{q}\times\mathbf{x}\right)/r\right) \rightarrow \varepsilon_{ijk}\partial_j\left(\varepsilon_{klm}q_l x_m/r\right)$

$= \underbrace{\varepsilon_{ijk}\varepsilon_{klm}}_{\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}} \quad q_l \underbrace{\partial_j\left(\frac{x_m}{r}\right)}_{(\partial_j x_m)\frac{1}{r}+x_m\left(\underbrace{\partial_j\frac{1}{r}}_{\frac{-1}{r^2}\partial_j r=\frac{-1}{r^2}\frac{x_j}{r}}\right)}$

$= \left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right) q_l\left[\frac{\partial_j x_m}{r}-\frac{x_m x_j}{r^3}\right]$

$= \delta_{il}\delta_{jm}q_l\frac{\partial_j x_m}{r}-\delta_{il}\delta_{jm}q_l\frac{x_m x_j}{r^3}-\delta_{im}\delta_{jl}q_l\frac{\partial_j x_m}{r}+\delta_{im}\delta_{jl}q_l\frac{x_m x_j}{r^3}$

$= q_i\underbrace{\frac{\partial_m x_m}{r}}_{\frac{1}{r}\delta_{mm}=\frac{3}{r}}-q_i\underbrace{\frac{x_m x_m}{r^3}}_{\frac{r^2}{r^3}=\frac{1}{r}}-q_j\underbrace{\frac{\partial_j x_i}{r}}_{\frac{1}{r}\delta_{ji}}+q_j\frac{x_i x_j}{r^3}$

$= \frac{3q_i}{r}-\frac{q_i}{r}-\frac{q_i}{r}+q_j\frac{x_i x_j}{r^3}$

$= \frac{q_i}{r}+\frac{q_j x_i x_j}{r^3}$

$\rightarrow \frac{\mathbf{q}}{r}+\frac{\mathbf{x}(\mathbf{q}\cdot\mathbf{x})}{r^3}$

2 Delta-Distribution

$I=\int_{-2}^{\infty}ds\int_{-\infty}^{2}dt\,\underbrace{\delta(2s+3t+6)}_{\frac{1}{2}\delta(s-(-\frac{3}{2}t-3))}\delta(6st)h(s,t).$

Integration über s: $s=-\frac{3}{2}t-3$;

$s\geq-2\rightarrow-\frac{3}{2}t-3\geq-2\rightarrow\frac{3}{2}t\leq-1\rightarrow t\leq-\frac{2}{3}$.

$I=\int_{-\infty}^{-2/3}dt\,\frac{1}{2}\delta\left(6\left(-\frac{3}{2}t-3\right)t\right)h(-\frac{3}{2}t-3,t).$

Ableitung: $\left(-9\left(t+2\right)t\right)'=-18t-18$

$I=\int_{-\infty}^{-2/3}dt\,\frac{1}{2}\quad\underbrace{\delta\left(-9\left(t+2\right)t\right)}_{\frac{1}{|-18t-18|}\delta(t+2)+\frac{1}{|-18t-18|}\delta(t)}\quad h(-\frac{3}{2}t-3,t).$

Da $t=0>-2/3$ bleibt nur Lösung $t=-2<-2/3$ übrig:

$I=\int_{-\infty}^{-2/3}dt\,\frac{1}{2}\underbrace{\frac{1}{|-18t-18|}\delta(t+2)}_{\frac{1}{18}\delta(t+2)}h(-\frac{3}{2}t-3,t).$

$I=\frac{1}{36}h(0,-2).$

3 Koordinatentransformation

a) Transformationsmatrix für kovariante Komponenten: ${a_i}^j = \frac{\partial x^j}{\partial x'^i} \rightarrow$

$$\begin{pmatrix} {a_1}^1 & {a_1}^2 \\ {a_2}^1 & {a_2}^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial r} & \frac{\partial (r\varphi)}{\partial r} \\ \frac{\partial r}{\partial \varphi} & \frac{\partial (r\varphi)}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} 1 & \varphi \\ 0 & r \end{pmatrix}.$$

Basisvektoren: $\mathbf{e}'_i = {a_i}^j \mathbf{e}_j$.

$\rightarrow \mathbf{e}'_1 = {a_1}^1 \mathbf{e}_1 + {a_1}^2 \mathbf{e}_2 = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \varphi \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \varphi \end{pmatrix}$.

$\mathbf{e}'_2 = {a_2}^1 \mathbf{e}_1 + {a_2}^2 \mathbf{e}_2 = 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + r \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$.

b) $g'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j \rightarrow \begin{pmatrix} g'_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{e}'_1 \cdot \mathbf{e}'_1 & \mathbf{e}'_1 \cdot \mathbf{e}'_2 \\ \mathbf{e}'_2 \cdot \mathbf{e}'_1 & \mathbf{e}'_2 \cdot \mathbf{e}'_2 \end{pmatrix} = \begin{pmatrix} 1+\varphi^2 & \varphi r \\ \varphi r & r^2 \end{pmatrix}$.

Nicht-orthogonal, da Nicht-Diagonalelemente auch besetzt.

c) Transformationsmatrix für kontravariante Komponenten: $a' = a^{-1}$. Invertieren von a:

$\left(\begin{array}{cc|cc} 1 & \varphi & 1 & 0 \\ 0 & r & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & \varphi & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{r} \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & -\frac{\varphi}{r} \\ 0 & 1 & 0 & \frac{1}{r} \end{array}\right)$.

${a'_i}^j = \begin{pmatrix} 1 & -\frac{\varphi}{r} \\ 0 & \frac{1}{r} \end{pmatrix}$.

Transformation:

$v'^j = {a'_i}^j v^i = \left(a'^T\right)^j{}_i v^i = \begin{pmatrix} 1 & 0 \\ -\frac{\varphi}{r} & \frac{1}{r} \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2}{r} \end{pmatrix}$.

d) Rücktransformation von kovarianten Vektorkomponenten:

$w'_i = (0,1) \rightarrow w'_i = {a_i}^j w_j \rightarrow w_j = {a'_j}^i w'_i$

Da Indizes i nebeneinander stehen, fassen wir w'_i als Spaltenvektor auf:

$w_j = \begin{pmatrix} 1 & -\frac{\varphi}{r} \\ 0 & \frac{1}{r} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\varphi}{r} \\ \frac{1}{r} \end{pmatrix}$.

Vereinfachen:

$\left(w_i \delta_j^k - g_{ij} w^k\right) A^{ij}{}_k = \left(w_i \delta_j^k - g_{ij} w^k\right) v^i v^j w_k$

$= w_i \delta_j^k v^i v^j w_k - g_{ij} w^k v^i v^j w_k$

$= w_i v^i v^k w_k - v_i v^i w^k w_k$

Ausrechnen im Euklidschen ungestrichenen System:

$= (\mathbf{w} \cdot \mathbf{v})^2 - \mathbf{v}^2 \mathbf{w}^2$

$= \left(-\frac{\varphi}{r} \times 0 + \frac{1}{r} \times 2\right)^2 - \left(2^2\right)\left(\frac{\varphi^2}{r^2} + \frac{1}{r^2}\right)$

$= \frac{4}{r^2} - 4\frac{\varphi^2}{r^2} - 4\frac{1}{r^2} = -4\frac{\varphi^2}{r^2}$.