3. Tutorium - Lösungen **16.11.2012**

3.1 Tensoren

a) Transformationsmatrix $\mathbf{e}'_i = a_i{}^j \mathbf{e}_j$: $a_1{}^1 = \cos\varphi$, $a_1{}^2 = \sin\varphi$, $a_2{}^1 = -\sin\varphi$, $a_2{}^2 = \cos\varphi$. (Notation: $c \equiv \cos\varphi$, $s \equiv \sin\varphi$)
$A'_{jk} = a_j{}^l a_k{}^m A_{lm}$.
$A'_{11} = a_1{}^l a_1{}^m A_{lm} = a_1{}^1 a_1{}^1 A_{11} + a_1{}^1 a_1{}^2 A_{12} + a_1{}^2 a_1{}^1 A_{21} + a_1{}^2 a_1{}^2 A_{22} = c^2 1 + cs1 + sc2 + ss3 = c^2 + 3sc + 3s^2$.
$A'_{12} = a_1{}^l a_2{}^m A_{lm} = a_1{}^1 a_2{}^1 A_{11} + a_1{}^1 a_2{}^2 A_{12} + a_1{}^2 a_2{}^1 A_{21} + a_1{}^2 a_2{}^2 A_{22} = -cs1 + c^2 1 - s^2 2 + sc3 = c^2 + 2sc - 2s^2$.
$A'_{21} = a_2{}^l a_1{}^m A_{lm} = a_2{}^1 a_1{}^1 A_{11} + a_2{}^1 a_1{}^2 A_{12} + a_2{}^2 a_1{}^1 A_{21} + a_2{}^2 a_1{}^2 A_{22} = -sc1 - s^2 1 + c^2 2 + cs3 = 2c^2 + 2sc - s^2$.
$A'_{22} = a_2{}^l a_2{}^m A_{lm} = a_2{}^1 a_2{}^1 A_{11} + a_2{}^1 a_2{}^2 A_{12} + a_2{}^2 a_2{}^1 A_{21} + a_2{}^2 a_2{}^2 A_{22} = +s^2 1 - sc1 - cs2 + c^2 3 = 3c^2 - 3sc + s^2$.
In Matrizenform geschrieben: $a_i{}^j \to a = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$
$A'_{jk} = a_j{}^l a_k{}^m A_{lm} \to A' = aAa^T = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} c+s & -s+c \\ 2c+3s & -2s+3c \end{pmatrix} =$
$\begin{pmatrix} c^2+3sc+3s^2 & c^2+2sc-2s^2 \\ 2c^2+2sc-s^2 & 3c^2-3sc+s^2 \end{pmatrix}$.
b) Metrischer Tensor: $g'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j$.
$g'_{11} = \mathbf{e}'_1 \cdot \mathbf{e}'_1 = \begin{pmatrix} c \\ s \end{pmatrix} \cdot \begin{pmatrix} c \\ s \end{pmatrix} = \cos^2\varphi + \sin^2\varphi = 1$.
$g'_{12} = \mathbf{e}'_1 \cdot \mathbf{e}'_2 = \begin{pmatrix} c \\ s \end{pmatrix} \cdot \begin{pmatrix} -s \\ c \end{pmatrix} = -cs + sc = 0$.
$g'_{21} = \mathbf{e}'_2 \cdot \mathbf{e}'_1 = \begin{pmatrix} -s \\ c \end{pmatrix} \cdot \begin{pmatrix} c \\ s \end{pmatrix} = -cs + sc = 0$.
$g'_{22} = \mathbf{e}'_2 \cdot \mathbf{e}'_2 = \begin{pmatrix} -s \\ c \end{pmatrix} \cdot \begin{pmatrix} -s \\ c \end{pmatrix} = \sin^2\varphi + \cos^2\varphi = 1$.
$g'_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
$g'^{ij} g'_{jk} = \delta_k{}^i \to g'^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
c) $A'^{11} = g'^{1k} g'^{1l} A'_{kl} = g'^{11} g'^{11} A'_{11} + g'^{11} g'^{12} A'_{12} + g'^{12} g'^{11} A'_{21} + g'^{12} g'^{12} A'_{22} = A'_{11} = c^2 + 3sc + 3s^2$.
Und ähnlich: $A'^{12} = A'_{12} = c^2 + 2sc - 2s^2$, $A'^{21} = A'_{21} = 2c^2 + 2sc - s^2$, $A'^{22} = A'_{22} = 3c^2 - 3sc + s^2$.
d) Transformationsmatrix $\mathbf{e}''_i = a_i{}^j \mathbf{e}_j$: $a_1{}^1 = 1$, $a_1{}^2 = 0$, $a_2{}^1 = -2$, $a_2{}^2 = 1$.
$A''_{jk} = a_j{}^l a_k{}^m A_{lm}$.
$A''_{11} = a_1{}^l a_1{}^m A_{lm} = a_1{}^1 a_1{}^1 A_{11} + a_1{}^1 a_1{}^2 A_{12} + a_1{}^2 a_1{}^1 A_{21} + a_1{}^2 a_1{}^2 A_{22} = 1 + 0 + 0 + 0 = 1$.
$A''_{12} = a_1{}^l a_2{}^m A_{lm} = a_1{}^1 a_2{}^1 A_{11} + a_1{}^1 a_2{}^2 A_{12} + a_1{}^2 a_2{}^1 A_{21} + a_1{}^2 a_2{}^2 A_{22} = -2 \times 1 + 1 + 0 + 0 = -1$.
$A''_{21} = a_2{}^l a_1{}^m A_{lm} = a_2{}^1 a_1{}^1 A_{11} + a_2{}^1 a_1{}^2 A_{12} + a_2{}^2 a_1{}^1 A_{21} + a_2{}^2 a_1{}^2 A_{22} = -2 \times 1 + 0 + 2 + 0 = 0$.
$A''_{22} = a_2{}^l a_2{}^m A_{lm} = a_2{}^1 a_2{}^1 A_{11} + a_2{}^1 a_2{}^2 A_{12} + a_2{}^2 a_2{}^1 A_{21} + a_2{}^2 a_2{}^2 A_{22} = 4 \times 1 - 2 \times 1 - 2 \times 2 + 3 = 1$.
e) $g_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j$.
$g''_{11} = \mathbf{e}''_1 \cdot \mathbf{e}''_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$.
$g''_{12} = \mathbf{e}''_1 \cdot \mathbf{e}''_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -2$.
$g''_{21} = \mathbf{e}''_2 \cdot \mathbf{e}''_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -2$.
$g''_{22} = \mathbf{e}''_2 \cdot \mathbf{e}''_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 5$.
$g''^{ij} g''_{jk} = \delta^i_k \to$Inverse bilden:

$\left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ -2 & 5 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 5 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right)$
$g''^{ij} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$
$A''^{11} = g''^{1k} g''^{1l} A''_{kl} = g''^{11} g''^{11} A''_{11} + g''^{11} g''^{12} A''_{12} + g''^{12} g''^{11} A''_{21} + g''^{12} g''^{12} A''_{22}$
$= 25 \times 1 + 10 \times (-1) + 10 \times 0 + 4 \times 1 = 25 - 10 + 0 + 1 = 19,$
$A''^{12} = g''^{1k} g''^{2l} A''_{kl} = 10 \times 1 + 5 \times (-1) + 4 \times 0 + 2 \times 1 = 10 - 5 + 0 + 2 = 7,$
$A''^{21} = g''^{2k} g''^{1l} A''_{kl} = 10 \times 1 + 4 \times (-1) + 5 \times 0 + 2 \times 1 = 10 - 4 + 0 + 2 = 8,$
$A''^{22} = g''^{2k} g''^{2l} A''_{kl} = 4 \times 1 + 2 \times (-1) + 2 \times 0 + 1 \times 1 = 4 - 2 + 0 + 1 = 3.$
f) Es kommt stets die gleiche Zahl heraus, da die Spur basisunabhängig ist:
$g^{ij} A_{ji} = g^{11} A_{11} + g^{12} A_{21} + g^{21} A_{12} + g^{22} A_{22} = 1 \times 1 + 0 \times 1 + 0 \times 2 + 1 \times 3 = 4.$
$g''^{ij} A''_{ji} = g''^{11} A''_{11} + g^{''12} A''_{21} + g''^{21} A''_{12} + g''^{22} A''_{22} = 5 \times 1 + 2 \times (-1) + 2 \times 0 + 1 \times 1 = 4.$
$A^{ij} A_{ji} = A^{11} A_{11} + A^{12} A_{21} + A^{21} A_{12} + A^{22} A_{22} = 1 \times 1 + 1 \times 2 + 2 \times 1 + 3 \times 3 = 14.$
$A''^{ij} A''_{ji} = A''^{11} A''_{11} + A''^{12} A''_{21} + A''^{21} A''_{12} + A''^{22} A''_{22} = 19 \times 1 + 7 \times 0 + 8 \times (-1) + 3 \times 1 = 14.$
$A^{ij} A_{ij} = A^{11} A_{11} + A^{12} A_{12} + A^{21} A_{21} + A^{22} A_{22} = 1 \times 1 + 1 \times 1 + 2 \times 2 + 3 \times 3 = 15.$
$A''^{ij} A''_{ij} = A''^{11} A''_{11} + A''^{12} A''_{12} + A''^{21} A''_{21} + A''^{22} A''_{22} = 1 \times 1 + 1 \times 1 + 2 \times 2 + 3 \times 3 = 15.$
g)
$s = g''^{ij} A''_{jk} g''^{km} \left(A''_{mi} - A''_{im}\right) = A''^{im} \left(A''_{mi} - A''_{im}\right) = A''^{im} A''_{mi} - A''^{im} A''_{im} = A^{im} A_{mi} - A^{im} A_{im}$
$= 14 - 15 = -1.$
$t = A''^{i}{}_{j} A''{}_{k}{}^{l} \left(\delta^j_l \delta^k_i - \delta^j_i \delta^k_l + g''^{jk} g''_{il}\right) = A''^{i}{}_{j} A''{}_{k}{}^{l} \delta^j_l \delta^k_i - A''^{i}{}_{j} A''{}_{k}{}^{l} \delta^j_i \delta^k_l + A''^{i}{}_{j} A''{}_{k}{}^{l} g''^{jk} g''_{il}$
$= A''^{i}{}_{l} A''{}_{i}{}^{l} - A''^{i}{}_{i} A''{}_{k}{}^{k} + A''^{ik} A''{}_{ki} = A''^{il} A''{}_{il} - g''^{ij} A''{}_{ji} g^{kl} A''{}_{kl} + A''^{ik} A''{}_{ki} = 15 - 16 + 14 = 13.$

3.2 Levi-Civita Symbol

a) $\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$
Einsetzen: falls i, j, k und k, l, m jeweils paarweise verschieden sind: In der Summe über k trägt links nur 1 Term bei.
Falls i, j, und k eine gerade Permutation von l, m, k ist ergibt sich $+1$, z.B.:
$i = 1, j = 2, l = 1, m = 2$: $\varepsilon_{ijk} \varepsilon_{klm} = 1 \cdot 1 = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} = 1 - 0 = 1.$
Falls i, j, und k eine ungerade Permutation von l, m, k ist ergibt sich -1, z.B.:
$i = 1, j = 2, l = 2, m = 1$: $\varepsilon_{ijk} \varepsilon_{klm} = 1 \cdot (-1) = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} = 0 - 1 = -1.$
(Zyklisches Vertauschen ändert weder links noch rechts etwas am Ergebnis).
Falls i und j ident sind, ergibt sich links 0 und rechts
$\delta_{il} \delta_{im} - \delta_{im} \delta_{il} = 0$ (ohne Einsteinsche Summenkonvention).
b) Für komplexe Vektoren verwenden wir: $\vec{a} \cdot \vec{b} = \left\langle \vec{a} \,\middle|\, \vec{b} \right\rangle = a_i^* b_i$, und $\left(\vec{a} \times \vec{b}\right)_i = \varepsilon_{ijk} a_j b_k$. Weiters: $|\vec{a}|^2 = \vec{a} \cdot \vec{a}$.
$|\vec{a} \cdot \vec{b}|^2 + |\vec{a} \times \vec{b}|^2 = \left(\vec{a} \cdot \vec{b}\right)^* \left(\vec{a} \cdot \vec{b}\right) + \left(\vec{a} \times \vec{b}\right)^* \cdot \left(\vec{a} \times \vec{b}\right) = \left(a_i^* b_i\right)^* a_j^* b_j + \varepsilon_{ijk} a_j^* b_k^* \varepsilon_{ilm} a_l b_m = a_i b_i^* a_j^* b_j + \left(\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}\right) a_j^* b_k^* a_l b_m = a_i b_i^* a_j^* b_j + a_j^* a_j b_k^* b_k - a_j^* b_k^* a_k b_j = a_j^* a_j b_k^* b_k = \left(\vec{a} \cdot \vec{a}\right) \left(\vec{b} \cdot \vec{b}\right) = |\vec{a}|^2 |\vec{b}|^2.$
Damit folgen auch gleich die beiden Ungleichungen, da für jedes Quadrat $|\ldots|^2 \geq 0$ gilt, z.B. $|\vec{a}|^2 |\vec{b}|^2 = |\vec{a} \cdot \vec{b}|^2 + |\vec{a} \times \vec{b}|^2 \geq |\vec{a} \cdot \vec{b}|^2$.

3.3 Differentialoperatoren

a) $\operatorname{div} \operatorname{rot} \mathbf{v} \to \partial_i \varepsilon_{ijk} \partial_j v_k = \underbrace{\varepsilon_{ijk}}_{\text{antisymmetrisch}} \underbrace{\partial_i \partial_j}_{\text{symmetrisch}} v_k = 0.$
b) $\operatorname{rot} \operatorname{rot} \mathbf{v} \to \varepsilon_{ijk} \partial_j \varepsilon_{klm} \partial_l v_m = \underbrace{\varepsilon_{ijk} \varepsilon_{klm}}_{\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}} \partial_j \partial_l v_m = \underbrace{\partial_j \partial_i v_j}_{\partial_i \partial_j v_j} - \partial_j \partial_j v_i \to \operatorname{grad} \operatorname{div} \mathbf{v} - \Delta \mathbf{v}.$
c) $\nabla \cdot \mathbf{x} \to \partial_i x_i = \delta_{ii} = 3.$
d) $\nabla r \to \partial_i r = \partial_i \sqrt{x_j x_j} = \frac{1}{2} \frac{1}{\sqrt{x_j x_j}} \left[\underbrace{\left(\partial_i x_j\right)}_{\delta_{ij}} x_j + x_j \underbrace{\left(\partial_i x_j\right)}_{\delta_{ij}} \right] = \frac{2 \delta_{ij} x_j}{2r} = \frac{x_i}{r}.$