5. Tutorium - Lösungen **30.11.2012**

5.1 Distributionen

a) Integral über y: $y=-x$:
$I=\int_{-\infty}^{\infty} dx\delta(2x^2+4x-30)f(x,-x)=\int_{-\infty}^{\infty} dx\delta\left(2\left(x-3\right)\left(x+5\right)\right)f(x,-x)$
Ableitung des Integranden: $4x+4$.
$I=\left.\frac{f(x,-x)}{|4x+4|}\right|_{x=3}+\left.\frac{f(x,-x)}{|4x+4|}\right|_{x=-5}=\frac{f(3,-3)}{16}+\frac{f(-5,5)}{16}=\frac{1}{16}\left[f(3,-3)+f(-5,5)\right]$.
b) Integral über t: $t=s-2$:
$J=\int_{3/2}^{\infty} ds\delta(s(s-2)-3)\cos(s+s-2)=\int_{3/2}^{\infty} ds\delta((s+1)(s-3))\cos(2s-2)$.
Ableitung des Integranden: $2s-2$. Nur Lösung $s=3>3/2$ trägt bei.
$J=\left.\frac{\cos(2s-2)}{|2s-2|}\right|_{s=3}=\frac{\cos 4}{4}$.

5.2 Indexschreibweise

a)$\mathrm{rot}\left(\mathbf{E}\times\mathbf{x}\right)\to\varepsilon_{ijk}\partial_j\left(\varepsilon_{klm}E_lx_m\right)=\underbrace{\varepsilon_{ijk}\varepsilon_{klm}}_{\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}}\underbrace{\partial_j\left(E_lx_m\right)}_{\left(\partial_jE_l\right)x_m+E_l\underbrace{\left(\partial_jx_m\right)}_{\delta_{jm}}}=\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\left[\left(\partial_jE_l\right)x_m+E_l\delta_{jm}\right]$

$=\delta_{il}\delta_{jm}\left(\partial_jE_l\right)x_m+\delta_{il}\delta_{jm}E_l\delta_{jm}-\delta_{im}\delta_{jl}\left(\partial_jE_l\right)x_m-\delta_{im}\delta_{jl}E_l\delta_{jm}=x_j\left(\partial_jE_i\right)+\underbrace{\delta_{jj}}_{3}E_i-\left(\partial_jE_j\right)x_i-E_i$

$=x_j\left(\partial_jE_i\right)-x_i\left(\partial_jE_j\right)+2E_i\to\left(\mathbf{x}\cdot\nabla\right)\mathbf{E}-\mathbf{x}\left(\nabla\cdot\mathbf{E}\right)+2\mathbf{E}$
b) $g^{ij}B_{ij}=g^{ij}\left(A_{ij}-\frac{1}{2}g_{ij}A^k{}_k\right)=A^i{}_i-\frac{1}{2}\underbrace{\delta^i_i}_{3}A^k{}_k=-\frac{1}{2}A^i{}_i$.

$B^{ij}B_{ij}=\left(A^{ij}-\frac{1}{2}g^{ij}A^k{}_k\right)\left(A_{ij}-\frac{1}{2}g_{ij}A^m{}_m\right)=A^{ij}A_{ij}-\frac{1}{2}A^i{}_iA^k{}_k-\frac{1}{2}A^i{}_iA^m{}_m+\frac{1}{4}\underbrace{\delta^i_i}_{3}A^k{}_kA^m{}_m=A^{ij}A_{ij}-\frac{1}{4}\left(A^k{}_k\right)^2$.
$B^{ij}B^{mn}\left(g_{im}g_{jn}-g_{in}g_{jm}\right)=B^{ij}B_{ij}-B^{ij}B_{ji}=\left[A^{ij}A_{ij}-\frac{1}{4}\left(A^k{}_k\right)^2\right]-\left[A^{ij}A_{ji}-\frac{1}{4}\left(A^k{}_k\right)^2\right]=A^{ij}A_{ij}-A^{ij}A_{ji}$.

5.3 Maxwell-Gleichungen

a) $\nabla_iE_i=4\pi\rho$,
$\nabla_iB_i=0$,
$\varepsilon_{klm}\nabla_lB_m=4\pi j_k+\frac{\partial}{\partial t}E_k$,
$\varepsilon_{klm}\nabla_lE_m=-\frac{\partial}{\partial t}B_k$,
$f_k=\rho E_k+\varepsilon_{klm}j_lB_m$.
b) $\mathrm{div}\mathbf{j}=\mathrm{div}\left(\frac{1}{4\pi}\mathrm{rot}\mathbf{B}-\frac{1}{4\pi}\frac{\partial}{\partial t}\mathbf{E}\right)$
$\to\nabla_kj_k=\nabla_k\left(\frac{1}{4\pi}\varepsilon_{klm}\nabla_lB_m-\frac{1}{4\pi}\frac{\partial}{\partial t}E_k\right)$
$=\frac{1}{4\pi}\underbrace{\varepsilon_{klm}\nabla_k\nabla_l}_{=0}B_m-\frac{1}{4\pi}\frac{\partial}{\partial t}\underbrace{\nabla_kE_k}_{4\pi\rho}$
$=-\frac{\partial}{\partial t}\rho$.
Also, Kontinuitätsgleichung: $\mathrm{div}\mathbf{j}+\frac{\partial}{\partial t}\rho=0$.
c) $\mathrm{div}\mathbf{B}=0\to\nabla_iB_i=0$.
$\to\nabla_i\left(\varepsilon_{ikm}\nabla_kA_m\right)=\underbrace{\varepsilon_{ikm}\nabla_i\nabla_k}_{=0}A_m=0$. Ok.
$\mathrm{rot}\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{B}\to\varepsilon_{klm}\nabla_lE_m=-\frac{\partial}{\partial t}B_k$

$\to \varepsilon_{klm}\nabla_l\left(-\nabla_m\phi - \frac{\partial}{\partial t}A_m\right) = -\frac{\partial}{\partial t}\left(\varepsilon_{klm}\nabla_l A_m\right)$
$\to -\underbrace{\varepsilon_{klm}\nabla_l\nabla_m}_{=0}\phi - \varepsilon_{klm}\frac{\partial}{\partial t}\nabla_l A_m = -\varepsilon_{klm}\frac{\partial}{\partial t}\nabla_l A_m$. Ok.

5.4 Transformation von Differentialoperatoren

a) Transformationsmatrix für kovariante Komponenten: $a_i^{\ j} = \frac{\partial x^j}{\partial x'^i} \to$

$$\begin{pmatrix} a_1^{\ 1} & a_1^{\ 2} \\ a_2^{\ 1} & a_2^{\ 2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \frac{\partial(r\cos\varphi)}{\partial r} & \frac{\partial(r\sin\varphi)}{\partial r} \\ \frac{\partial(r\cos\varphi)}{\partial \varphi} & \frac{\partial(r\sin\varphi)}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{pmatrix}.$$

Basisvektoren: $\mathbf{e}'_i = a_i^{\ j}\mathbf{e}_j$.

$\to \mathbf{e}'_1 = a_1^{\ 1}\mathbf{e}_1 + a_1^{\ 2}\mathbf{e}_2 = \cos\varphi\begin{pmatrix}1\\0\end{pmatrix} + \sin\varphi\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$.

$\mathbf{e}'_2 = a_2^{\ 1}\mathbf{e}_1 + a_2^{\ 2}\mathbf{e}_2 = -r\sin\varphi\begin{pmatrix}1\\0\end{pmatrix} + r\cos\varphi\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-r\sin\varphi\\ r\cos\varphi\end{pmatrix}$.

b)

$$g'_{ij} = \mathbf{e}'_i\cdot\mathbf{e}'_j \to \begin{pmatrix} g'_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{e}'_1\cdot\mathbf{e}'_1 & \mathbf{e}'_1\cdot\mathbf{e}'_2 \\ \mathbf{e}'_2\cdot\mathbf{e}'_1 & \mathbf{e}'_2\cdot\mathbf{e}'_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\varphi + \sin^2\varphi & \cos\varphi\,(-r\sin\varphi) + \sin\varphi r\cos\varphi \\ -r\sin\varphi\cos\varphi + r\cos\varphi\sin\varphi & r^2\sin^2\varphi + r^2\cos^2\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

Orthogonal, da nur Diagonalelemente besetzt sind. (Für $r = 0$ ist Metrik singulär, da Determinante verschwindet $\leftrightarrow$ nicht invertierbar.)

c) Transformationsmatrix für kontravariante Komponenten: $a' = a^{-1}$. Invertieren von a:

$$\left(\begin{array}{cc|cc} \cos\varphi & \sin\varphi & 1 & 0 \\ -r\sin\varphi & r\cos\varphi & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & \frac{\sin\varphi}{\cos\varphi} & \frac{1}{\cos\varphi} & 0 \\ 0 & \frac{1}{\cos\varphi\sin\varphi} & \frac{1}{\cos\varphi} & \frac{1}{r\sin\varphi} \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & \cos\varphi & -\frac{1}{r}\sin\varphi \\ 0 & 1 & \sin\varphi & \frac{1}{r}\cos\varphi \end{array}\right).$$

Transformation: $v'^j = a'_i{}^{j} v^i = \left(a'^T\right)^j_{\ i} v^i = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\frac{1}{r}\sin\varphi & \frac{1}{r}\cos\varphi \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\sin\varphi\\ \frac{1}{r}\cos\varphi\end{pmatrix}$.

d) Zum selber Zeichnen (bitte nachfragen, wenn es dazu Fragen gibt!).

Basisvektoren kartesisch: $\{\mathbf{e}_1, \mathbf{e}_2\} = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$. $v^i = \begin{pmatrix}0\\1\end{pmatrix}$.

In Polarkoordinaten, z.B. an der Stelle $x'^i = \begin{pmatrix}2\\1\end{pmatrix}$: $x^i = \begin{pmatrix}2\cos 1\\ 2\sin 1\end{pmatrix} \approx \begin{pmatrix}1{,}08\\ 1{,}68\end{pmatrix}$

$\{\mathbf{e}'_1, \mathbf{e}'_2\} = \left\{\begin{pmatrix}\cos 1\\ \sin 1\end{pmatrix}, \begin{pmatrix}-2\sin 1\\ 2\cos 1\end{pmatrix}\right\} \approx \left\{\begin{pmatrix}0{,}54\\ 0{,}84\end{pmatrix}, \begin{pmatrix}-1{,}86\\ 1{,}08\end{pmatrix}\right\}$; $v'^i = \begin{pmatrix}\sin 1\\ \frac{1}{2}\cos 1\end{pmatrix} \approx \begin{pmatrix}0{,}84\\ 0{,}27\end{pmatrix}$.

e) $\mathbf{e}^j = a_i^{\ j}\mathbf{e}'^i = \left(a^T\right)^j_{\ i}\mathbf{e}'^i$.

$$\to \begin{pmatrix}\mathbf{e}^x\\ \mathbf{e}^y\end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}\begin{pmatrix}\mathbf{e}^r\\ \mathbf{e}^\varphi\end{pmatrix} = \begin{pmatrix} \cos\varphi\mathbf{e}^r - r\sin\varphi\mathbf{e}^\varphi \\ \sin\varphi\mathbf{e}^r + r\cos\varphi\mathbf{e}^\varphi \end{pmatrix}.$$

Die Ableitung transformiert genau wie die Basisvektoren mit a' zurück:

$$\frac{\partial}{\partial x^i} = \frac{\partial x'^j}{\partial x^i}\frac{\partial}{\partial x'^j} = a'_i{}^{j}\frac{\partial}{\partial x'^j} \quad \to \quad \begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} = \begin{pmatrix} \cos\varphi & -\frac{1}{r}\sin\varphi \\ \sin\varphi & \frac{1}{r}\cos\varphi \end{pmatrix}\begin{pmatrix}\partial_r\\ \partial_\varphi\end{pmatrix} = \begin{pmatrix} \cos\varphi\partial_r - \frac{1}{r}\sin\varphi\partial_\varphi \\ \sin\varphi\partial_r + \frac{1}{r}\cos\varphi\partial_\varphi \end{pmatrix}.$$

Kombiniert ergibt das:

$\nabla\phi(x,y) = \left(\partial_x\phi(x,y)\right)\mathbf{e}^x + \left(\partial_y\phi(x,y)\right)\mathbf{e}^y =$

$= \left(\cos\varphi\partial_r\phi(r,\varphi) - \frac{1}{r}\sin\varphi\partial_\varphi\phi(r,\varphi)\right)\left(\cos\varphi\mathbf{e}^r - r\sin\varphi\mathbf{e}^\varphi\right) + \left(\sin\varphi\partial_r\phi(r,\varphi) + \frac{1}{r}\cos\varphi\partial_\varphi\phi(r,\varphi)\right)\left(\sin\varphi\mathbf{e}^r + r\cos\varphi\mathbf{e}^\varphi\right)$

$= \left(\cos^2\varphi\partial_r\phi - \frac{1}{r}\sin\varphi\cos\varphi\partial_\varphi\phi + \sin^2\varphi\partial_r\phi + \frac{1}{r}\cos\varphi\sin\varphi\partial_\varphi\phi\right)\mathbf{e}^r + \left(0\times\partial_r\phi + \sin^2\varphi\partial_\varphi\phi + \cos^2\varphi\partial_\varphi\phi\right)\mathbf{e}^\varphi$

$= \left(\partial_r\phi(r,\varphi)\right)\mathbf{e}^r + \left(\partial_\varphi\phi(r,\varphi)\right)\mathbf{e}^\varphi$.

Nicht-orthogonale Parabelkoordinaten

f) Transformationsmatrix für kovariante Komponenten: ${a_i}^j = \frac{\partial x^j}{\partial x'^i} \rightarrow$

$$\begin{pmatrix} {a_1}^1 & {a_1}^2 \\ {a_2}^1 & {a_2}^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \frac{\partial \left(r\left(1-\varphi^2/2\right)\right)}{\partial r} & \frac{\partial (r\varphi)}{\partial r} \\ \frac{\partial \left(r\left(1-\varphi^2/2\right)\right)}{\partial \varphi} & \frac{\partial (r\varphi)}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} 1-\frac{\varphi^2}{2} & \varphi \\ -r\varphi & r \end{pmatrix}.$$

Basisvektoren: $\mathbf{e}'_i = {a_i}^j \mathbf{e}_j$.

$\rightarrow \mathbf{e}'_1 = {a_1}^1 \mathbf{e}_1 + {a_1}^2 \mathbf{e}_2 = \left(1-\frac{\varphi^2}{2}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \varphi \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{\varphi^2}{2} \\ \varphi \end{pmatrix}$.

$\mathbf{e}'_2 = {a_2}^1 \mathbf{e}_1 + {a_2}^2 \mathbf{e}_2 = -r\varphi \begin{pmatrix} 1 \\ 0 \end{pmatrix} + r \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -r\varphi \\ r \end{pmatrix}$.

g)

$$g'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j \rightarrow \begin{pmatrix} g'_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{e}'_1 \cdot \mathbf{e}'_1 & \mathbf{e}'_1 \cdot \mathbf{e}'_2 \\ \mathbf{e}'_2 \cdot \mathbf{e}'_1 & \mathbf{e}'_2 \cdot \mathbf{e}'_2 \end{pmatrix}$$

$$= \begin{pmatrix} \left(1-\frac{\varphi^2}{2}\right)^2 + \varphi^2 & \left(1-\frac{\varphi^2}{2}\right)(-r\varphi) + \varphi r \\ -r\varphi\left(1-\frac{\varphi^2}{2}\right) + r\varphi & r^2\varphi^2 + r^2 \end{pmatrix} = \begin{pmatrix} 1+\frac{\varphi^4}{4} & r\frac{\varphi^3}{2} \\ r\frac{\varphi^3}{2} & r^2\left(1+\varphi^2\right) \end{pmatrix}.$$

Nicht-orthogonal, da nicht nur Diagonalelemente besetzt sind. (Orthogonal nur für $\varphi = 0$, $r > 0$. Für $r = 0$ ist Metrik singulär.)

h) Transformationsmatrix für kontravariante Komponenten: $a' = a^{-1}$. Invertieren von a:

$$a^{-1} = \frac{1}{1+\frac{\varphi^2}{2}} \begin{pmatrix} 1 & -\frac{\varphi}{r} \\ \varphi & \frac{1}{r}\left(1-\frac{\varphi^2}{2}\right) \end{pmatrix}.$$

Transformation: $v'^j = {a'_i}^j v^i = \left(a'^T\right)^j{}_i v^i = \frac{1}{1+\frac{\varphi^2}{2}} \begin{pmatrix} 1 & \varphi \\ -\frac{\varphi}{r} & \frac{1}{r}\left(1-\frac{\varphi^2}{2}\right) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{1+\frac{\varphi^2}{2}} \begin{pmatrix} \varphi \\ \frac{1}{r}\left(1-\frac{\varphi^2}{2}\right) \end{pmatrix}$.

i) Zum selber Zeichnen (bitte nachfragen, wenn es dazu Fragen gibt!).

j) $\mathbf{e}^j = {a_i}^j \mathbf{e}'^i = \left(a^T\right)^j{}_i \mathbf{e}'^i$.

$$\rightarrow \begin{pmatrix} \mathbf{e}^x \\ \mathbf{e}^y \end{pmatrix} = \begin{pmatrix} 1-\frac{\varphi^2}{2} & -r\varphi \\ \varphi & r \end{pmatrix} \begin{pmatrix} \mathbf{e}^r \\ \mathbf{e}^\varphi \end{pmatrix} = \begin{pmatrix} \left(1-\frac{\varphi^2}{2}\right)\mathbf{e}^r - r\varphi\mathbf{e}^\varphi \\ \varphi\mathbf{e}^r + r\mathbf{e}^\varphi \end{pmatrix}.$$

Die Ableitung transformiert genau wie die Basisvektoren mit a' zurück:

$$\frac{\partial}{\partial x^i} = \frac{\partial x'^j}{\partial x^i}\frac{\partial}{\partial x^j} = {a'_i}^j \frac{\partial}{\partial x^j} \rightarrow \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \frac{1}{1+\frac{\varphi^2}{2}} \begin{pmatrix} 1 & -\frac{\varphi}{r} \\ \varphi & \frac{1}{r}\left(1-\frac{\varphi^2}{2}\right) \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\varphi \end{pmatrix} = \frac{1}{1+\frac{\varphi^2}{2}} \begin{pmatrix} \partial_r - \frac{\varphi}{r}\partial_\varphi \\ \varphi\partial_r + \frac{1}{r}\left(1-\frac{\varphi^2}{2}\right)\partial_\varphi \end{pmatrix}.$$

Kombiniert ergibt das:

$\nabla\phi(x,y) = \left[\mathbf{e}^x\partial_x + \mathbf{e}^y\partial_y\right]\phi(x,y) =$

$= \frac{1}{1+\frac{\varphi^2}{2}}\left[\left(\left(1-\frac{\varphi^2}{2}\right)\mathbf{e}^r - r\varphi\mathbf{e}^\varphi\right)\left(\partial_r - \frac{\varphi}{r}\partial_\varphi\right) + \left(\varphi\mathbf{e}^r + r\mathbf{e}^\varphi\right)\left(\varphi\partial_r + \frac{1}{r}\left(1-\frac{\varphi^2}{2}\right)\partial_\varphi\right)\right]\phi(r,\varphi)$

$= \frac{1}{1+\frac{\varphi^2}{2}}\left[\mathbf{e}^r\left(\left(1-\frac{\varphi^2}{2}\right)\partial_r - \left(1-\frac{\varphi^2}{2}\right)\frac{\varphi}{r}\partial_\varphi + \varphi^2\partial_r + \frac{\varphi}{r}\left(1-\frac{\varphi^2}{2}\right)\partial_\varphi\right)\right.$

$\left. \qquad + \mathbf{e}^\varphi\left(-r\varphi\partial_r + \varphi^2\partial_\varphi + r\varphi\partial_r + \left(1-\frac{\varphi^2}{2}\right)\partial_\varphi\right)\right]\phi(r,\varphi)$

$= \frac{1}{1+\frac{\varphi^2}{2}}\left[\mathbf{e}^r\left(\left(1+\frac{\varphi^2}{2}\right)\partial_r + 0\times\partial_\varphi\right) + \mathbf{e}^\varphi\left(0\times\partial_r + \left(1+\frac{\varphi^2}{2}\right)\partial_\varphi\right)\right]\phi(r,\varphi)$

$= \left[\mathbf{e}^r\partial_r + \mathbf{e}^\varphi\partial_\varphi\right]\phi(r,\varphi)$.