7. Tutorium - Lösungen **21.12.2012**

7.1 Laplace-Gleichung in Zylinderkoordinaten

Zylinderkoordinaten: $\mathbf{x} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$.

$\mathbf{e}'_j = \frac{\partial x^i}{\partial x'^j}\mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial x'^j}$.

$\rightarrow \mathbf{e}_r = \frac{\partial}{\partial r}\mathbf{x} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$, $\mathbf{e}_\varphi = \frac{\partial}{\partial \varphi}\mathbf{x} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}$, $\mathbf{e}_z = \frac{\partial}{\partial z}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Berechnung der Metrik:

$g'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j \rightarrow \left[g'_{ij}\right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, z.B. $\mathbf{e}_r \cdot \mathbf{e}_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = 1$, etc.

Determinante: $|g'| = \det\left[g'_{ij}\right] = 1 \times r^2 \times 1 = r^2$.

Inverse Metrik: $g'^{ij}g'_{jk} = \delta^i_k \rightarrow \left[g'_{ij}\right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Laplace-Operator: $\Delta = \frac{1}{\sqrt{|g'|}}\partial'_i \left(\sqrt{|g'|}g'^{ij}\partial'_j\right)$

$= \frac{1}{\sqrt{r^2}}\partial'_i \left(\sqrt{r^2}g'^{ij}\partial'_j\right)$

$= \frac{1}{r}\partial_r\left(r \times 1 \times \partial_r\right) + \frac{1}{r}\partial_\varphi\left(r \times \frac{1}{r^2} \times \partial_\varphi\right) + \frac{1}{r}\partial_z\left(r \times 1 \times \partial_z\right)$

$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial z}\frac{\partial}{\partial z}$

$= \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$.

b) Homogene Laplace-Gleichung: $\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}\right)\Phi(r,\varphi,z) = 0$.

Ansatz: $\Phi(r,\varphi,z) = \Phi_1(r)\Phi_2(\varphi)\Phi_3(z)$.

$\left[\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right)\Phi_1(r)\right]\Phi_2(\varphi)\Phi_3(z) + \frac{1}{r^2}\Phi_1(r)\left[\frac{\partial^2}{\partial\varphi^2}\Phi_2(\varphi)\right]\Phi_3(z) + \Phi_1(r)\Phi_2(\varphi)\left[\frac{\partial^2}{\partial z^2}\Phi_3(z)\right] = 0$.

$\left[\frac{1}{\Phi_1(r)}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right)\Phi_1(r)\right] + \frac{1}{r^2}\frac{1}{\Phi_2(\varphi)}\left[\frac{\partial^2}{\partial\varphi^2}\Phi_2(\varphi)\right] = -\frac{1}{\Phi_3(z)}\left[\frac{\partial^2}{\partial z^2}\Phi_3(z)\right] = A(r,\varphi) = A(z) = A = const.$

$\rightarrow \Phi_3''(z) = -A\Phi_3(z)$.

$r^2\left[\frac{1}{\Phi_1(r)}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right)\Phi_1(r) - A\right] = -\frac{1}{\Phi_2(\varphi)}\left[\frac{\partial^2}{\partial\varphi^2}\Phi_2(\varphi)\right] = B(r) = B(\varphi) = B = const.$

$\rightarrow \Phi_2''(\varphi) = -B\Phi_2(\varphi)$.

$\rightarrow \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right)\Phi_1(r) = A\Phi_1(r) + \frac{B}{r^2}\Phi_1(r)$.

7.2 Greensche Funktion

a) $\mathcal{L}_x = -\frac{d^2}{dx^2} - \lambda$, $f(x) = \alpha x$, $\mathcal{L}_x y(x) = f(x)$.

Ansatz: $G(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(k)e^{ik(x-x')}dk$

und $\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik(x-x')}dk$ einsetzen in

$\mathcal{L}_x G(x,x') = \delta(x-x')$,

$\left(-\frac{d^2}{dx^2} - \lambda\right)G(x,x') = \delta(x-x')$,

$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(k)\left(-\frac{d^2}{dx^2} - \lambda\right)e^{ik(x-x')}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik(x-x')}dk$,

$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(k)\left(k^2 - \lambda\right)e^{ik(x-x')}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik(x-x')}dk$.

Vergleich der Integranden:

$\tilde{G}(k)\left(k^2-\lambda\right)=1. \quad \rightarrow \quad \tilde{G}(k)=\frac{1}{k^2-\lambda}=\frac{1}{\left(k-\sqrt{\lambda}\right)\left(k+\sqrt{\lambda}\right)}$.

(Etwas ausführlicher: Fourier-Transformation $\int_{-\infty}^{\infty} dx\, e^{-ik'(x-x')}$ auf beiden Seiten angewandt:

$\rightarrow \int_{-\infty}^{\infty} dx\, e^{-ik'(x-x')}\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\tilde{G}(k)\left(k^2-\lambda\right)e^{ik(x-x')}=\int_{-\infty}^{\infty} dx\, e^{-ik'(x-x')}\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(x-x')}$.

$\rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\tilde{G}(k)\left(k^2-\lambda\right)\underbrace{\int_{-\infty}^{\infty} dx\, e^{i(k-k')(x-x')}}_{2\pi\delta(k-k')}=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\underbrace{\int_{-\infty}^{\infty} dx\, e^{i(k-k')(x-x')}}_{2\pi\delta(k-k')}$.

$\rightarrow \tilde{G}(k')\left(k'^2-\lambda\right)=1.)$

$G(x,x')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik(x-x')}}{\left(k-\sqrt{\lambda}\right)\left(k+\sqrt{\lambda}\right)}dk$.

Beide Pole liegen genau auf der reellen Achse. Um den Residuensatz anwenden zu können, werden die Pole leicht verschoben, z.B. nach oben: $(k-k_1)(k-k_2)=\left(k+\sqrt{\lambda}-i\varepsilon\right)\left(k-\sqrt{\lambda}-i\varepsilon\right)$ mit $k_1=-\sqrt{\lambda}+i\varepsilon$, $k_2=\sqrt{\lambda}+i\varepsilon$. Beide Pole liegen nun im oberen Bereich. Für $x-x'>0$: Großkreis oben schließen ($ik=i(\mathrm{Re}k+i\mathrm{Im}k)=i\mathrm{Re}k-\mathrm{Im}k$: Für $\mathrm{Im}k>0$ exponentiell gedämpft). Für $x-x'<0$: Großkreis unten schließen: hier sind keine Pole mehr eingeschlossen.

$G(x,x')=H(x-x')\left[2\pi i\mathrm{Res}_{k\rightarrow-\sqrt{\lambda}+i\varepsilon}\frac{1}{2\pi}\frac{e^{ik(x-x')}}{\left(k+\sqrt{\lambda}-i\varepsilon\right)\left(k-\sqrt{\lambda}-i\varepsilon\right)}+2\pi i\mathrm{Res}_{k\rightarrow-\sqrt{\lambda}+i\varepsilon}\frac{1}{2\pi}\frac{e^{ik(x-x')}}{\left(k+\sqrt{\lambda}-i\varepsilon\right)\left(k-\sqrt{\lambda}-i\varepsilon\right)}\right]+$
$H(x'-x)\times 0$

$=H(x-x')2\pi i\left[\lim_{k\rightarrow-\sqrt{\lambda}+i\varepsilon}(k+\sqrt{\lambda}-i\varepsilon)\frac{1}{2\pi}\frac{e^{ik(x-x')}}{\left(k+\sqrt{\lambda}-i\varepsilon\right)\left(k-\sqrt{\lambda}-i\varepsilon\right)}-\lim_{k\rightarrow\sqrt{\lambda}+i\varepsilon}(k+i\omega)\frac{1}{2\pi}\frac{e^{ik(x-x')}}{\left(k+\sqrt{\lambda}-i\varepsilon\right)\left(k-\sqrt{\lambda}-i\varepsilon\right)}\right]$

$=H(x-x')\left[i\frac{e^{i\left(-\sqrt{\lambda}+i\varepsilon\right)(x-x')}}{-2\sqrt{\lambda}}+i\frac{e^{i\left(\sqrt{\lambda}+i\varepsilon\right)(x-x')}}{2\sqrt{\lambda}}\right]\overset{\varepsilon\rightarrow0}{=}H(x-x')\frac{1}{2i\sqrt{\lambda}}\left(e^{-i\sqrt{\lambda}(x-x')}-e^{i\sqrt{\lambda}(x-x')}\right)=H(x-x')\frac{-\sin\left(\sqrt{\lambda}(x-x')\right)}{\sqrt{\lambda}}=:G_I(x,x')$.

b) Randbedingung: $G(0,x'>0)=0$ ist erfüllt.

$y(x)=\int_0^d G(x,x')f(x')dx'$

(Falls nicht, kann man homogene Greensche Funktionen hinzufügen, um die Randbedingungen zu erfüllen, z.B. für $0<x'<d$:

$G(x,x')=G_I(x,x')+A\sin\left(\sqrt{\lambda}(x-x')\right)+B\cos\left(\sqrt{\lambda}(x-x')\right)$.

$G(0,x')=0+A\sqrt{\lambda}\sin\left(\sqrt{\lambda}(x-x')\right)+B\sqrt{\lambda}\cos\left(\sqrt{\lambda}(x-x')\right)=0.$ (I)

$G'(x,x')=\delta(x-x')\frac{-\sin\left(\sqrt{\lambda}(x-x')\right)}{\sqrt{\lambda}}-H(x-x')\cos\left(\sqrt{\lambda}(x-x')\right)$
$+A\sqrt{\lambda}\cos\left(\sqrt{\lambda}(x-x')\right)-B\sqrt{\lambda}\sin\left(\sqrt{\lambda}(x-x')\right)$.

$G'(0,x')=0-0+A\sqrt{\lambda}\cos\left(\sqrt{\lambda}(x-x')\right)-B\sqrt{\lambda}\sin\left(\sqrt{\lambda}(x-x')\right)=0.$ (II)

Aus (I) und (II) folgt: $A=0$, $B=0$.

$\rightarrow G(x,x')=G_I(x,x')$.)

Lösung für $0<x<d$:

$y(x)=\int_0^d G(x,x')f(x')dx'=\underbrace{\int_0^d H(x-x')}_{\int_0^x}\frac{-\sin\left(\sqrt{\lambda}(x-x')\right)}{\sqrt{\lambda}}\alpha x'dx'$

$=\frac{\alpha}{\sqrt{\lambda}}\int_0^x\sin\left(\sqrt{\lambda}(x'-x)\right)x'dx'$

$=\frac{\alpha}{\sqrt{\lambda}}\left[-\frac{1}{\sqrt{\lambda}}\cos\left(\sqrt{\lambda}(x'-x)\right)x'\Big|_{x'=0}^{x}+\int_0^x\frac{1}{\sqrt{\lambda}}\cos\left(\sqrt{\lambda}(x'-x)\right)dx'\right]$

$=\frac{\alpha}{\sqrt{\lambda}}\left[-\frac{1}{\sqrt{\lambda}}\cos\left(\sqrt{\lambda}\times 0\right)x+0+\frac{1}{\lambda}\sin\left(\sqrt{\lambda}\times 0\right)-\frac{1}{\lambda}\sin\left(\sqrt{\lambda}(-x)\right)\right]$

$=\frac{\alpha}{\sqrt{\lambda}}\left[-\frac{1}{\sqrt{\lambda}}x+\frac{1}{\lambda}\sin\left(\sqrt{\lambda}x\right)\right]=-\frac{\alpha}{\lambda}x+\frac{\alpha}{\lambda\sqrt{\lambda}}\sin\left(\sqrt{\lambda}x\right)$.

Probe: $y(0)=0$,

$y'(0)=-\frac{\alpha}{\lambda}+\frac{\alpha}{\lambda}\cos(0)=0$.

$\frac{d}{dx}y(x)=-\frac{\alpha}{\lambda}+\frac{\alpha}{\lambda}\cos\left(\sqrt{\lambda}x\right)$.

$\frac{d^2}{dx^2}y(x)=-\frac{\alpha}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}x\right)$.

$\mathcal{L}_x y(x) = -\frac{d^2}{dx^2} y(x) - \lambda y(x)$
$= -\frac{\alpha}{\sqrt{\lambda}} \sin\left(\sqrt{\lambda}x\right) - \lambda\left[-\frac{\alpha}{\lambda}x + \frac{\alpha}{\lambda\sqrt{\lambda}} \sin\left(\sqrt{\lambda}x\right)\right]$
$= \alpha x = f(x).$

7.3 Sturm-Liouville-Problem

a)$\mathcal{L}_x y(x) = y'' + \frac{1}{x}y' - \frac{B}{x^2}y \stackrel{!}{=} a_0(x)y(x) + a_1(x)\frac{d}{dx}y(x) + a_2(x)\frac{d^2}{dx^2}y(x).$
$a_0(x) = -\frac{B}{x^2}$, $a_1(x) = \frac{1}{x}$, $a_2(x) = 1$.
$\rightarrow$Transformation auf Sturm-Liouville Form $\mathcal{S}_x y(x) = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] y(x) + q(x)y(x)$
mit $p(x) = e^{\int \frac{a_1(x)}{a_2(x)}dx} = \exp\left(\int \frac{1}{x}dx\right) = \exp\left(\ln x + c\right) = x \exp c = x\tilde{c}.$
$q(x) = p(x)\frac{a_0(x)}{a_2(x)} = -x\tilde{c}\frac{B}{x^2} = -\frac{\tilde{c}B}{x}.$
$\mathcal{S}_x y(x) = \frac{d}{dx}\left(x\tilde{c}\frac{d}{dx}\right) y(x) - \frac{\tilde{c}B}{x}y(x).$
Im folgenden: $\tilde{c} = 1$: $\mathcal{L}_x y(x) = \frac{a_2(x)}{p(x)}\mathcal{S}_x y(x) = \frac{1}{x}\left[\frac{d}{dx}\left(x\frac{d}{dx}\right) y(x) - \frac{B}{x}y(x)\right] = \frac{1}{x}\left[(xy')' - \frac{By}{x}\right].$
<u>Probe</u>: $\frac{1}{x}\left[(xy')' - \frac{By}{x}\right] = \frac{1}{x}\left[y' + xy'' - \frac{By}{x}\right] = y'' + \frac{1}{x}y' - \frac{B}{x^2}y = \mathcal{L}_x y(x).$
b) $\mathcal{L}_x y(x) = f(x) = -\lambda y \quad \rightarrow \quad \mathcal{S}_x y(x) = F(x),$
$F(x) = p(x)\frac{f(x)}{a_2(x)} = x\frac{-\lambda y}{1} = -x\lambda y.$
$\rho(x) = \frac{F(x)}{-\lambda y(x)} = x.$
$\left[\mathcal{S}_x + \lambda\rho(x)\right] y(x) = 0 \quad \leftrightarrow \quad \left[\frac{d}{dx}\left(x\frac{d}{dx}\right) - \frac{B}{x} + x\lambda\right] y(x) = 0$ für $x \in [a,b]$.
also $\rho(x) = x$, $p(x) = x$
Sturm-Liouville Transformation:
$\xi = t(x) = \int_a^x \sqrt{\frac{\rho(s)}{p(s)}}ds = \int_a^x \sqrt{\frac{s}{s}}ds = \int_a^x 1ds = x - a \quad \rightarrow \quad x(t) = t + a.$
$w(t) = \sqrt[4]{p(x(t))\rho(x(t))}y(x(t)) = \sqrt[4]{(t+a)\,(t+a)}y(t+a) = \sqrt{t+a}y(t+a).$
$\hat{q}(t) = \frac{1}{\rho}\left[-q - \sqrt[4]{p\rho}\left(p\left(\frac{1}{\sqrt[4]{p\rho}}\right)'\right)'\right] = \frac{1}{x}\left[\frac{B}{x} - \sqrt[4]{x^2}\left(x\left(\frac{1}{\sqrt[4]{x^2}}\right)'\right)'\right]$
$\quad = \frac{1}{x}\left[\frac{B}{x} - \sqrt{x}\left(x\left(\frac{1}{\sqrt{x}}\right)'\right)'\right] = \frac{1}{x}\left[\frac{B}{x} - \sqrt{x}\left(x\frac{-1}{2}x^{-3/2}\right)'\right] = \frac{1}{x}\left[\frac{B}{x} - \sqrt{x}\left(\frac{1}{4}\frac{1}{x^{3/2}}\right)\right]$
$\quad = \frac{1}{x}\left[\frac{B}{x} - \frac{1}{4}\frac{1}{x}\right] = \frac{1}{x^2}\left(B - \frac{1}{4}\right) = \frac{1}{(t+a)^2}\left(B - \frac{1}{4}\right).$
$c = t(b) = b - a.$
Liouvillesche Normalform:
$-\frac{d^2}{dt^2}w(t) + \left[\hat{q}(t) - \lambda\right] w(t) = 0$ für $t \in [0, b-a]$,
$-\frac{d^2}{dt^2}w(t) + \left[\frac{1}{(t+a)^2}\left(B - \frac{1}{4}\right) - \lambda\right] w(t) = 0$ für $t \in [0, b-a]$.
c) <u>Probe</u>: $w(t) = \sqrt{t+a}y(t+a)$.
$\frac{d^2}{dt^2}w(t) = \frac{d^2}{dt^2}\left(\sqrt{t+a}y(t+a)\right) = \frac{d}{dt}\left(\frac{1}{2}\frac{1}{\sqrt{t+a}}y(t+a) + \sqrt{t+a}y'(t+a)\right)$
$= -\frac{1}{4}\frac{1}{(t+a)^{3/2}}y(t+a) + \frac{1}{2}\frac{1}{\sqrt{t+a}}y'(t+a) + \frac{1}{2}\frac{1}{\sqrt{t+a}}y'(t+a) + \sqrt{t+a}y''(t+a)$
$= -\frac{1}{4}\frac{1}{x^{3/2}}y(x) + \frac{1}{x^{1/2}}x'(x) + x^{1/2}y''(x)$
$\rightarrow -\frac{d^2}{dt^2}w(t) + \left[\frac{1}{(t+a)^2}\left(B - \frac{1}{4}\right) - \lambda\right] w(t) = -\sqrt{x}y''(x) - \frac{y'(x)}{\sqrt{x}} + \frac{1}{4}\frac{y(x)}{x^{3/2}} + \frac{y(x)}{x^{3/2}}\left(B - \frac{1}{4}\right) - \lambda\sqrt{x}y = 0.$
$\stackrel{/(-\sqrt{x})}{\rightarrow} y'' + \frac{y'}{x} - \frac{By}{x^2} + \lambda y = 0 = \mathcal{L}_x y + \lambda y.$
d) $B = \frac{1}{4}$: $-\frac{d^2}{dt^2}w(t) + \left[-\lambda\right] w(t) = 0$. $\rightarrow w'' = -\lambda w.$
Ansatz: $w(t) = \alpha\sin(\sqrt{\lambda}t) + \beta\cos(\sqrt{\lambda}t).$
$y(1) = y(2) = 0 \rightarrow w(0) = w(1) = 0 \rightarrow \beta = 0,\ \sqrt{\lambda} = \pi m$ mit $m \in \mathbb{N}$.
$\rightarrow\lambda = (\pi m)^2$, $w(t) = \alpha\sin(\pi m t).$
Eingesetzt in $y(x) = y(t+a) = \frac{w(t)}{\sqrt{t+a}} = \frac{w(x-a)}{\sqrt{x}}$: $y(x) = \frac{1}{\sqrt{x}}w(x-a) = \frac{1}{\sqrt{x}}\alpha\sin(\pi m(x-a)).$
<u>Probe</u>: $y'(x) = -\frac{1}{2}\frac{\alpha}{x^{3/2}}\sin(\pi m(x-a)) + \frac{\alpha\pi m}{\sqrt{x}}\cos(\pi m(x-a))$
$y''(x) = \frac{3}{4}\frac{\alpha}{x^{5/2}}\sin(\pi m(x-a)) - \frac{1}{2}\frac{\alpha\pi m}{x^{3/2}}\cos(\pi m(x-a))$
$\qquad -\frac{1}{2}\frac{\alpha\pi m}{x^{3/2}}\cos(\pi m(x-a) - \frac{\alpha(\pi m)^2}{\sqrt{x}}\sin(\pi m(x-a))$

$$\to y'' + \frac{1}{x}y' - \frac{1}{4}\frac{1}{x^2}y = \sin(\pi m(x-a))\left[\frac{3}{4}\frac{\alpha}{x^{5/2}} - \frac{\alpha(\pi m)^2}{\sqrt{x}} - \frac{1}{x}\frac{1}{2}\frac{\alpha}{x^{3/2}} - \frac{1}{4}\frac{1}{x^2}\frac{\alpha}{\sqrt{x}}\right]$$
$$+ \cos(\pi m(x-a))\left[-\frac{1}{2}\frac{\alpha\pi m}{x^{3/2}} - \frac{1}{2}\frac{\alpha\pi m}{x^{3/2}} + \frac{1}{x}\frac{\alpha\pi m}{\sqrt{x}}\right]$$
$$= \sin(\pi m(x-a))\left[-\frac{\alpha(\pi m)^2}{\sqrt{x}}\right] + 0 = -\left(\pi m\right)^2 y(x) = -\lambda y(x). \to \text{OK.}$$