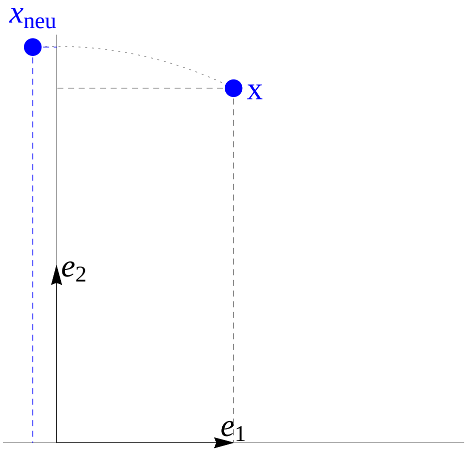
**1. Tutorium - Lösungen** **11.10.2013**

## 1.1 Multiple Choice Fragen

a) $\left[\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\right]^2 = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 6 & 11 \end{pmatrix}$.
b) $\sum_{i=1}^{3} x_i^2 = (x_1)^2 + (x_2)^2 + (x_3)^2 = 1^2 + (-1)^2 + 2^2 = 6$.
c) $a_{10} + 2a_{11} + 3a_{22} = 0 + 2 \times 1 + 3 \times 1 = 5$.
d) $\sum_{m=1}^{3}\sum_{n=1}^{3} (a_{mn})^2 = \sum_{m=1}^{3}\left[(a_{m1})^2 + (a_{m2})^2 + (a_{m3})^2\right] = a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{22}^2 + a_{23}2 + a_{32}^2 + a_{33}^2$
$= 0^2 + (-1)^2 + (-1)^2 + 1^2 + 0^2 + (-1)^2 + 1^2 + 1^2 + 0^2 = 6$.
e) $\sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = 2 \times 1 + 4 \times 0 + 6 \times (-1) = -4$.
f) $e^{2ix} = 1 + 2ix + \frac{1}{2}(2ix)^2 + \frac{1}{3!}(2ix)^3 + \ldots$ Gesuchter Term: $\frac{8}{6} i^3 x^3 = -\frac{4}{3} i x^3$.
g) Allgemeine Taylor-Reihe: $f(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0)$.
$f'(x) = -\sin(\sin(x))\cos(x)$, $f''(x) = -\cos(\sin(x))\cos^2(x) + \sin(\sin(x))\sin(x)$,
$f'''(x) = \sin(\sin(x))\cos^3(x) + \cos(\sin(x))2\cos(x)\sin(x) + \cos(\sin(x))\cos(x)\sin(x) + \sin(\sin(x))\cos(x)$.
$f'''(0) = 0 \rightarrow$ Term der dritten Ordnung verschwindet.
Alternativ:
$\cos(\sin x) = 1 - \frac{1}{2}\sin^2 x + \frac{1}{4!}\sin^4 x + \ldots = 1 - \frac{1}{2}\left(x - \frac{1}{3!}x^3 + \ldots\right)^2 + \ldots = 1 - \frac{1}{2}x^2 + O(x^4)$ Nach dem Ausquadrieren kommen nur gerade Potenzen von $x$ vor $\rightarrow$ es gibt keinen Term von der Ordnung $O(x^3)$.
h)Partiell integrieren: $\int_0^1 x e^{2x} dx = x\frac{1}{2}e^{2x}\Big|_0^1 - \int_0^1 \frac{1}{2}e^{2x} dx = \frac{e^2}{2} - 0 - \frac{1}{4}e^{2x}\Big|_0^1 = \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4}$.
Alternativ: $\int_0^1 x e^{2x} dx = \frac{d}{d\alpha}\int_0^1 e^{\alpha x} dx\Big|_{\alpha=2} = \frac{d}{d\alpha}\frac{e^{\alpha x}}{\alpha}\Big|_{x=0}^{1}\Big|_{\alpha=2} = \frac{d}{d\alpha}\frac{e^{\alpha}-1}{\alpha}\Big|_{\alpha=2} = \frac{e^{\alpha}\alpha - (e^{\alpha}-1)}{\alpha^2}\Big|_{\alpha=2} = \frac{2e^2 - e^2 + 1}{4}$
$= \frac{e^2}{4} + \frac{1}{4}$.
i) $\frac{d}{dx}\tan(x^2) = \frac{d}{dx}\frac{\sin(x^2)}{\cos(x^2)} = \frac{\cos(x^2)\frac{d}{dx}\sin(x^2) - \sin(x^2)\frac{d}{dx}\cos(x^2)}{[\cos(x^2)]^2} = \frac{\cos(x^2)\cos(x^2)2x + \sin(x^2)\sin(x^2)2x}{[\cos(x^2)]^2} = \frac{2x}{[\cos(x^2)]^2}$.
j) $\frac{d}{d\alpha}\frac{d}{d\beta}[\cos(\alpha+\beta)\sin(\alpha-\beta)] = \frac{d}{d\alpha}[-\sin(\alpha+\beta)\sin(\alpha-\beta) - \cos(\alpha+\beta)\cos(\alpha-\beta)]$
$= -\cos(\alpha+\beta)\sin(\alpha-\beta) - \sin(\alpha+\beta)\cos(\alpha-\beta) + \sin(\alpha+\beta)\cos(\alpha-\beta) + \cos(\alpha+\beta)\sin(\alpha-\beta) = 0$.
Alternativ: erkenne, dass $-\sin(\alpha+\beta)\sin(\alpha-\beta) - \cos(\alpha+\beta)\cos(\alpha-\beta) = -\cos((\alpha+\beta) - (\alpha-\beta))$
$= -\cos(2\beta)$, daher $\frac{d}{d\alpha}(-\cos(2\beta)) = 0$.

### 1.2 Aktive und passive Drehung

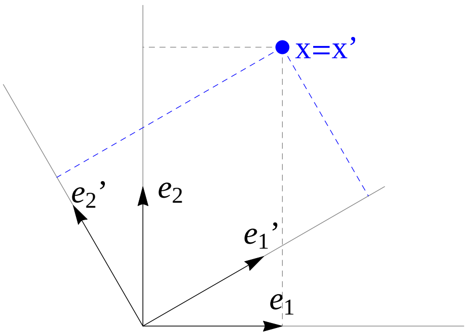
a)




b) $R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$

$$\mathbf{x}_{\text{neu}} = R(\alpha)\mathbf{x} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} - 1 \\ \frac{1}{2} + \sqrt{3} \end{pmatrix} \approx \begin{pmatrix} -0.13 \\ 2.23 \end{pmatrix}.$$

c)




d) $S(\alpha) = R(\alpha)$.

$$\mathbf{e}_1' = S\,\mathbf{e}_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \approx \begin{pmatrix} 0.87 \\ 0.5 \end{pmatrix}.$$

$$\mathbf{e}_2' = S\,\mathbf{e}_2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \approx \begin{pmatrix} -0.5 \\ 0.87 \end{pmatrix}.$$

$R$ und $S$ sind ident.

e) $T(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$

$$\mathbf{x}' = T\,\mathbf{x} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + \frac{\sqrt{3}}{2} \\ \sqrt{3} - \frac{1}{2} \end{pmatrix} \approx \begin{pmatrix} 1.87 \\ 1.23 \end{pmatrix}$$

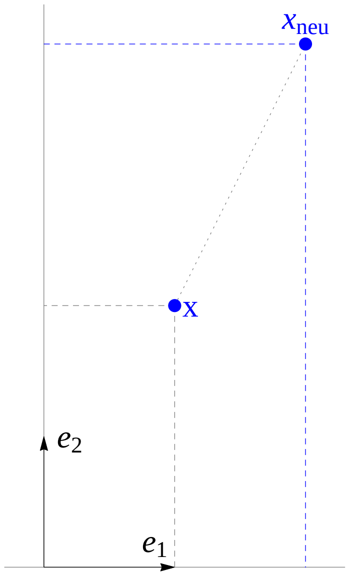
$R$ und $T$ sind inverse zueinander. (Sie sind auch transponiert zueinander).

f) $R(\alpha)T(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

### 1.3 Aktive und passive Skalierung

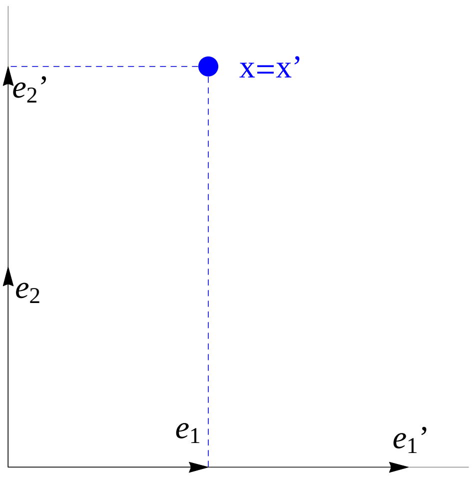
a)




b) $S(\lambda) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

$\mathbf{x}_{\text{neu}} = S(\lambda)\mathbf{x} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

c)




d) $U(\lambda) = S(\lambda)$.

$\mathbf{e}_1' = U\,\mathbf{e}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

$\mathbf{e}_2' = U\,\mathbf{e}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

e) $V(\alpha) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

$\mathbf{x}' = V\,\mathbf{x} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$

f) $S(\lambda)V(\lambda) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 \times \frac{1}{2} & 0 \\ 0 & 2 \times \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

g) $l = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \sqrt{1+4} = \sqrt{5}$ .

$l' = \sqrt{\mathbf{x}'^T\mathbf{x}'} = \sqrt{\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}} = \sqrt{\frac{1}{4}+1} = \frac{\sqrt{5}}{2}$.

$g(\lambda) = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}$

$\rightarrow \qquad l = \sqrt{\mathbf{x}'^T\, g(\lambda)\, \mathbf{x}'} = \sqrt{\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 2^2 & 0 \\ 0 & 2^2 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}} = \sqrt{\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix}} = \sqrt{5}$.