2. Test - Lösungen **23.1.2015**

1 Greensche Funktion [30 Punkte]

a)[10] Gleichung für die Greensche Funktion : $\left(\frac{d^2}{dx^2} - 2\frac{d}{dx} + 2\right) G(x,x') = \delta(x-x')$

Fourier-Entwicklung : $G(x,x') = (2\pi)^{-1} \int_{-\infty}^{\infty} \tilde{G}(k)\, e^{ik(x-x')} dk$

Ersetzung in der Gleichung : $\frac{1}{2\pi}\int_{-\infty}^{\infty} (-k^2 - 2ik + 2)\tilde{G}(k)\, e^{ik(x-x')} dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')} dk$

Fourier-Transformierte Greensche Funktion : $\tilde{G}(k) = -\frac{1}{k^2 + 2ik - 2}$

b)[20] Fourier-Transformation : $G_I(x,x') = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik(x-x')}}{k^2+2ik-2} dk = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik(x-x')}}{(k+1+i)(k-1+i)} dk$

Zwei Pole liegen in der unteren Halbebene der komplexen Zahlen und kein Pol liegt in der oberen Halbebene. Mit Hilfe des Residuensatzes (Integrationspfad : oberer Halbkreis für $x > x'$ und unterer Halbkreis für $x < x'$) ergibt sich

$$
\begin{aligned}
G_I(x,x') &= H(x'-x)\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik(x-x')}}{(k+1+i)(k-1+i)} dk = H(x'-x)\left(i \lim_{k\to 1-i} \frac{e^{ik(x-x')}}{k+1+i} + i \lim_{k\to -1-i} \frac{e^{ik(x-x')}}{k-1+i}\right) \\
&= H(x'-x)\left(i\frac{e^{i(1-i)(x-x')}}{2} - i\frac{e^{i(-1-i)(x-x')}}{2}\right) = H(x'-x)\frac{i}{2}e^{x-x'}\left(e^{i(x-x')} - e^{-i(x-x')}\right) \\
&= -H(x'-x)e^{x-x'}\sin(x-x')
\end{aligned}
$$

Randbedingungen : $G_I(0,x') = -H(x')e^{-x'}\sin(-x') \neq 0$

$G_I'(0,x') = \delta(x')e^{-x'}\sin(-x') - H(x')e^{-x'}\sin(-x') - H(x')e^{-x'}\cos(-x') \neq 0$

Homogene Greensche Funktion : z.B. $G_0(x,x') = e^{x-x'}\sin(x-x')$

$\frac{d}{dx}G_0(x,x') = e^{x-x'}\sin(x-x') + e^{x-x'}\cos(x-x')$, $\frac{d^2}{dx^2}G_0(x,x') = 2e^{x-x'}\cos(x-x') \to \mathcal{L}_x G_0(x,x') = 0$

$G(x,x') = G_I(x,x') + G_0(x,x') = (1 - H(x'-x))e^{x-x'}\sin(x-x')$

Randbedingungen : $G(0,x') = (1-H(x'))e^{-x'}\sin(-x') = 0$ wenn $x' > 0$.

$G'(0,x') = \delta(x')e^{-x'}\sin(-x') - (1-H(x'))e^{-x'}\sin(-x') - (1-H(x'))e^{-x'}\cos(-x') = 0$ wenn $x > 0$

2 Differentialgleichung [30 Punkte]

a)[6] Separationsansatz : $\Psi(x,y) = u(r)v(\theta)$

Differentialgleichung :

$$
\begin{aligned}
& \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{2}{r^2}\right) u(r)v(\theta) = -\lambda u(r)v(\theta) \\
\longrightarrow\quad & v(\theta)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u(r)}{\partial r}\right) + \frac{u(r)}{r^2}\frac{\partial^2 v(\theta)}{\partial\theta^2} + \frac{2}{r^2}u(r)v(\theta) = -\lambda u(r)v(\theta) \\
\longrightarrow\quad & \frac{r}{u(r)}\frac{\partial}{\partial r}\left(r\frac{\partial u(r)}{\partial r}\right) + \frac{1}{v(\theta)}\frac{\partial^2 v(\theta)}{\partial\theta^2} = -\lambda r^2 - 2 \\
\longrightarrow\quad & \frac{r}{u(r)}\frac{\partial}{\partial r}\left(r\frac{\partial u(r)}{\partial r}\right) + \lambda r^2 = -\frac{1}{v(\theta)}\frac{\partial^2 v(\theta)}{\partial\theta^2} - 2 = A \\
\longrightarrow\quad & r\frac{\partial}{\partial r}\left(r\frac{\partial u(r)}{\partial r}\right) + \lambda r^2 u(r) - Au(r)\,, \quad \frac{\partial^2 v(\theta)}{\partial\theta^2} = -(A+2)v(\theta) \\
\longrightarrow\quad & r^2\frac{\partial^2 u(r)}{\partial r^2} + r\frac{\partial u(r)}{\partial r} + (\lambda r^2 - A)u(r) = 0\,, \quad \frac{\partial^2 v(\theta)}{\partial\theta^2} = -(A+2)v(\theta)\,.
\end{aligned}
$$

b)[6] Differentialgleichung : $u''(r) + \frac{1}{r}u'(r) + (\lambda - A/r^2)u(r) = 0$.

$\alpha_0 = \lim_{r\to 0} r\frac{1}{r} = 1$, $\beta_0 = \lim_{r\to 0} r^2(\lambda - A/r^2) = -A \longrightarrow x = 0$ ist ein regulärer singulärer Punkt.

c)[18] Ansatz (Entwicklung am regulären singulären Punkt $x = 0$) : $u(r) = \sum_{k=0}^{\infty} w_k r^{k+\sigma}$
$u'(r) = \sum_{k=0}^{\infty}(k+\sigma)w_k r^{k+\sigma-1}$
$u''(r) = \sum_{k=0}^{\infty}(k+\sigma)(k+\sigma-1)w_k r^{k+\sigma-2}$
Ersetzung des Ansatzes in die Differentialgleichung:

$$\begin{aligned}
& \sum_{k=0}^{\infty}(k+\sigma)(k+\sigma-1)w_k r^{k+\sigma} + \sum_{k=0}^{\infty}(k+\sigma)w_k r^{k+\sigma} + \sum_{k=0}^{\infty} w_k r^{k+\sigma+2} - \sum_{k=0}^{\infty} w_k r^{k+\sigma} = 0 \\
\longrightarrow\quad & \sum_{k=0}^{\infty}(k+\sigma)(k+\sigma-1)w_k r^{k+\sigma} + \sum_{k=0}^{\infty}(k+\sigma)w_k r^{k+\sigma} + \sum_{k=2}^{\infty} w_{k-2} r^{k+\sigma} - \sum_{k=0}^{\infty} w_k r^{k+\sigma} = 0 \\
\longrightarrow\quad & (\sigma(\sigma-1)+\sigma-1)w_0 x^{\sigma} + (\sigma(\sigma+1)+\sigma+1-1)w_1 x^{\sigma+1} \\
& + \sum_{k=2}^{\infty}\left[((k+\sigma)(k+\sigma-1)+(k+\sigma)-1)w_k + \lambda w_{k-2}\right] r^{k+\sigma} = 0 \\
\longrightarrow\quad & (\sigma^2-1)w_0 r^{\sigma} + ((\sigma+1)^2-1)w_1 r^{\sigma+1} + \sum_{k=2}^{\infty}\left[((k+\sigma)^2-1)w_k + w_{k-2}\right] r^{k+\sigma} = 0 \\
\longrightarrow\quad & (\sigma^2-1)w_0 = 0\,, \quad ((\sigma+1)^2-1)w_1 = 0\,, \quad ((k+\sigma)^2-1)w_k = -w_{k-2}
\end{aligned}$$

Weil $w_0 = 1$ und $\sigma > 0$ gilt $\sigma = 1$ aus der Bedingung $(\sigma^2-1)w_0 = 0$.
$((\sigma+1)^2-1)w_1 = 3w_1 = 0 \to w_1 = 0$
Rekursionsrelation : $w_k = -\frac{1}{k(k+2)}w_{k-2} \to w_k = 0$ für ungerade k.
Wenn $k = 2n$,

$$w_{2n} \;=\; -\frac{1}{4n(n+1)}w_{2(n-1)} = \left(-\frac{1}{4}\right)^2 \frac{1}{n(n-1)}\frac{1}{(n+1)n}w_{2(n-2)} = \cdots = \left(-\frac{1}{4}\right)^n \frac{1}{n!}\frac{1}{(n+1)!}$$

3 Multiple Choice Fragen [40 Punkte, 4 Punkte je Frage]

1) $\sqrt{\frac{a}{\pi}}\int_{-\infty}^{\infty} x^4 e^{-ax^2}dx = \sqrt{\frac{a}{\pi}}\int_{-\infty}^{\infty}(t/a)^2 e^{-t}\frac{1}{2\sqrt{at}}dt = \frac{1}{2a^2\sqrt{\pi}}\int_{-\infty}^{\infty} t^{3/2}e^{-t}dt = \frac{1}{a^2\sqrt{\pi}}\int_0^{\infty} t^{3/2}e^{-t}dt = \frac{1}{a^2\sqrt{\pi}}\Gamma(5/2) = \frac{1}{a^2\sqrt{\pi}}(3/2)\Gamma(3/2) = \frac{1}{a^2\sqrt{\pi}}(3/2)(1/2)\Gamma(1/2) = \frac{3}{4a^2}$
2) $\frac{2n+1}{2}\int_0^{\pi} P_n(\cos\theta)(13-12\cos\theta)^{-1/2}\sin\theta d\theta = \frac{2n+1}{2}\int_{-1}^{1} P_n(x)(13-12x)^{-1/2}dx = \frac{2n+1}{2}\int_{-1}^{1} P_n(x)(3^2+2^2-2\cdot 3\cdot 2x)^{-1/2}dx = \frac{2n+1}{6}\int_{-1}^{1} P_n(x)((2/3)^2-2(2/3)x)^{-1/2}dx = \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{2^n}{3^{n+1}}$

Sturm-Liouville-Problem
$\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)y(x) = \lambda\rho(x)y(x) \to p(x)\frac{d^2}{dx^2}y(x) + p'(x)\frac{d}{dx}y(x) = \lambda\rho(x)y(x)$
Vergleich mit der angegebenen Differentialgleichung : $p'(x)/p(x) = a_1(x)/a_2(x)$ und $\lambda\rho(x)/p(x) = f(x)/a_2(x)$
Integration der 1. Gleichung nach x : $\log p(x) = \int \frac{a_1(x)}{a_2(x)}dx \longrightarrow p(x) = e^{\int \frac{a_1(x)}{a_2(x)}dx}$.
$\lambda\rho(x)y(x) = p(x)\frac{f(x)}{a_2(x)} = \frac{f(x)}{a_2(x)}e^{\int \frac{a_1(x)}{a_2(x)}dx}$
3) $p_A(x) = e^{\int \frac{-2x}{1-x^2}dx} = e^{\log(1-x^2)} = 1-x^2$
alternative Lösung : Wenn $p_A(x) = 1-x^2$, $\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) = (1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}$.
4) $\rho_A(x) = \frac{1}{\lambda y(x)}\frac{\lambda y(x)}{1-x^2}(1-x^2) = 1$
5) $p_B(x) = e^{-2\int x dx} = e^{-x^2}$.
alternative Lösung : Wenn $p_A(x) = e^{-x^2}$, $\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) = e^{-x^2}\left(\frac{d^2}{dx^2} - 2x\frac{d}{dx}\right)$.
6) $\rho_B(x) = \frac{1}{\lambda y(x)}\lambda y(x)e^{-x^2} = e^{-x^2}$

7) $\Gamma(4) = 3! = 6$
8) $\Gamma(5/2) = (3/2)\Gamma(3/2) = (3/4)\Gamma(1/2) = (3/4)\sqrt{\pi}$
9) $|\Gamma(1+i)|^2 = \Gamma(1+i)\Gamma(1-i) = i\Gamma(i)\Gamma(1-i) = \frac{i\pi}{\sin(i\pi)} = \frac{2\pi}{e^{\pi}-e^{-\pi}}$.
10) $L_0^1(x) = (-1)^0\frac{1!}{1!0!0!}x^0 = 1$.