11. Tutorium - Lösungen **9.1.2015**

- ANMERKUNG: Jeder ist selber für den sinnvollen Umgang mit Lösungszetteln verantwortlich. Letztendlich geht es darum, was man selber lernt und versteht.

11.1 Multiple Choice Fragen

a)

$$\begin{aligned}|(ix)!|^2 &= |\Gamma(1+ix)|^2 = \Gamma(1+ix)\Gamma(1-ix) = \Gamma(1+ix)\left((-ix)\Gamma(-ix)\right) = -ix\Gamma(-ix)\Gamma(1+ix) \\ &= -ix\frac{\pi}{\sin(i\pi x)} = \frac{\pi x}{\sinh(\pi x)}\end{aligned}$$

b)

$$\begin{aligned}\Gamma\left(n+\frac{1}{2}\right) &= \frac{2n-1}{2}\Gamma\left(n-\frac{1}{2}\right) = \frac{2n-1}{2}\frac{2n-3}{2}\Gamma\left(n-\frac{3}{2}\right) = \frac{2n-1}{2}\frac{2n-3}{2}\frac{2n-5}{2}\Gamma\left(n-\frac{5}{2}\right) \\ &= \cdots = \frac{2n-1}{2}\frac{2n-3}{2}\frac{2n-5}{2}\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}\end{aligned}$$

c)

$$\langle v^n\rangle = \int |\vec{v}|^n \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-m(v_x^2+v_y^2+v_z^2)/(2kT)} d^3v = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty v^{n+2}e^{-mv^2/(2kT)}dv$$

Variablentransformation : $t = (m/(2kT))v^2 \to dv = \sqrt{kT/(2m)}\ t^{-1/2}\,dt$

$$\begin{aligned}\langle v^n\rangle &= 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\left(\frac{2kT}{m}\right)^{n/2+1}\left(\frac{kT}{2m}\right)^{1/2}\int_0^\infty t^{n/2+1}e^{-t}t^{-1/2}dt \\ &= \frac{2}{\sqrt{\pi}}\left(\frac{2kT}{m}\right)^{n/2}\int_0^\infty t^{(n+1)/2}e^{-t}dt = \frac{2}{\sqrt{\pi}}\left(\frac{2kT}{m}\right)^{n/2}\Gamma\left(\frac{n+3}{2}\right)\end{aligned}$$

d)

$$\int_0^\infty \frac{E^{1/2}}{e^{\beta E}-1}dE = \int_0^\infty E^{1/2}\frac{e^{-\beta E}}{1-e^{-\beta E}}dE = \underbrace{\int_0^\infty E^{1/2}\sum_{n=1}^\infty e^{-n\beta E}dE}_{\text{Variablentransformation }:t=n\beta E} = \sum_{n=1}^\infty\int_0^\infty\left(\frac{t}{n\beta}\right)^{1/2}e^{-t}\frac{1}{n\beta}dt$$

$$= \beta^{-3/2}\sum_{n=1}^\infty n^{-3/2}\int_0^\infty t^{1/2}e^{-t}dt = \beta^{-3/2}\zeta\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right) \simeq 1.3\sqrt{\pi}\beta^{-3/2}$$

Weitere Information : $\zeta(t) = \sum_{n=1}^\infty n^{-t}$ ist bekannt als Zeta-Funktion.
e) Greensche Funktion : $\mathcal{L}_x G(x,x') = \delta(x,x')$
Fouriertransformation : $G(x,x') = (2\pi)^{-1}\int_{-\infty}^\infty \tilde{G}(k)e^{i(x-x')k}dk$
Differentialgleichung : $(2\pi)^{-1}\int_{-\infty}^\infty (ik-\alpha)(ik+\bar{\alpha})\tilde{G}(k)e^{i(x-x')k}dk = (2\pi)^{-1}\int_{-\infty}^\infty e^{i(x-x')k}dk$

$$\longrightarrow \tilde{G}(k) = \frac{1}{(ik-\alpha)(ik+\bar{\alpha})} = -\frac{1}{(k+i\alpha)(k-i\bar{\alpha})}$$

f) Wenn $x - x' > 0$

$$\frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{ik(x-x')}}{(k+ia-b)(k-ia-b)}dk = i\mathrm{Res}_{k\to b+ia}\frac{e^{ik(x-x')}}{(k+ia-b)(k-ia-b)} = i\lim_{k\to b+ia}\frac{e^{ik(x-x')}}{k+ia-b}$$
$$= i\frac{e^{i(b+ia)(x-x')}}{2ia} = \frac{1}{2a}e^{ib(x-x')}e^{-a(x-x')}$$

Wenn $x - x' < 0$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik(x-x')}}{(k+ia-b)(k-ia-b)}dk = -i\mathrm{Res}_{k\to b-ia}\frac{e^{ik(x-x')}}{(k+ia-b)(k-ia-b)} = -i\lim_{k\to b-ia}\frac{e^{ik(x-x')}}{k-ia-b}$$
$$= -i\frac{e^{i(b-ia)(x-x')}}{-2ia} = \frac{1}{2a}e^{ib(x-x')}e^{a(x-x')}$$

Das Integral entspricht der Greenschen Funktion im Bsp.e mit $\alpha = a + ib$.

$$G(x,x') = \frac{1}{2a}e^{ib(x-x')}e^{-a|x-x'|}$$

11.2 Gamma- und Beta-Funktionen

a) R-abhängigkeit

$$V_n(R) = \int H\left(R^2 - \sum_{i=1}^n x_i^2\right)d^n x = \int H\left(R^2\left(1-\sum_{i=1}^n \frac{x_i^2}{R^2}\right)\right)R^n d^n\left(\frac{x}{R}\right) = R^n\int H\left(1-\sum_{i=1}^n \tilde{x}_i^2\right)d^n\tilde{x} = R^n V_n(1)$$

wobei $\tilde{x}_i = x_i/R$.
Alternative Lösung (ohne Heaviside-Funktion) :

$$V_n(R) = \int_{x_1^2+x_2^2+\cdots+x_n^2<R^2} d^n x = R^n\int_{\tilde{x}_1^2+\tilde{x}_2^2+\cdots+\tilde{x}_n^2<1} d^n\tilde{x} = R^n V_n(1)$$

b) Das Volumen der n-dimensionalen Kugel ist das Integral des Volumens der $(n-1)$-dimensionalen Kugel mit Radius $\sqrt{R^2-x^2}$.

$$V_n(R) = \int_{-R}^{R} V_{n-1}\left(\sqrt{R^2-x^2}\right)dx = V_{n-1}(1)\int_{-R}^{R}\left(R^2-x^2\right)^{(n-1)/2}dx = \underbrace{R^n V_{n-1}(1)}_{=RV_{n-1}(R)}\int_{-1}^{1}\left(1-\tilde{x}^2\right)^{(n-1)/2}d\tilde{x}$$

Variablentransformation : $t = \tilde{x}^2 \to d\tilde{x} = (1/2)t^{-1/2}dt$.

$$\int_{-1}^{1}\left(1-\tilde{x}^2\right)^{(n-1)/2}d\tilde{x} = 2\int_{0}^{1}\left(1-\tilde{x}^2\right)^{(n-1)/2}d\tilde{x} = \int_0^1 (1-t)^{(n-1)/2}t^{-1/2}dt = B\left(\frac{1}{2},\frac{n+1}{2}\right)$$

Rekursionsrelation :

$$V_n(R) = RB\left(\frac{1}{2},\frac{n+1}{2}\right)V_{n-1}(R)$$

c) Mit Gamma-Funktionen ist die Rekursionsrelation

$$V_n(R) = R\frac{\Gamma(1/2)\Gamma((n+1)/2)}{\Gamma(n/2+1)}V_{n-1}(R)$$

Rekursionsrelation zwischen V_n und V_{n-2}

$$\begin{aligned} V_n(R) &= \sqrt{\pi}R\frac{\Gamma((n+1)/2)}{\Gamma(n/2+1)}V_{n-1}(R) = \pi R^2\frac{\Gamma((n+1)/2)\Gamma(n/2)}{\Gamma(n/2+1)\Gamma(n/2+1/2)}V_{n-2}(R) \\ &= \pi R^2\frac{\Gamma(n/2)}{\Gamma(n/2+1)}V_{n-2}(R) = \pi R^2\frac{2}{n}V_{n-2}(R) \end{aligned}$$

Für gerades $n = 2k$

$$\begin{aligned} V_{2k}(R) &= \pi R^2\frac{1}{k}V_{2(k-1)}(R) = \pi^2R^4\frac{1}{k(k-1)}V_{2(k-2)}(R) = \pi^3R^6\frac{1}{k(k-1)(k-2)}V_{2(k-3)}(R) \\ &= \cdots = \pi^{k-1}R^{2(k-1)}\frac{1}{k(k-1)(k-2)\cdots 2}V_2(R) = \frac{\pi^k}{k!}R^{2k} = \frac{\pi^{n/2}}{(n/2)!}R^n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}R^n \end{aligned}$$

Für ungerades $n = 2k+1$

$$\begin{aligned}
V_{2k+1}(R) &= \pi R^2 \frac{2}{2k+1} V_{2k-1}(R) = \pi^2 R^4 \frac{2^2}{(2k+1)(2k-1)} V_{2k-3)}(R) = \pi^3 R^6 \frac{2^3}{(2k+1)(2k-1)(2k-3)} V_{2k-5}(R) \\
&= \cdots = (2\pi)^{k-1} R^{2(k-1)} \frac{1}{(2k+1)(2k-1)(2k-3)\cdots 5} V_3(R) = 2(2\pi)^k R^{2k+1} \frac{1}{(2k+1)!!} \\
&= 2(2\pi)^{(n-1)/2} R^n \frac{1}{n!!} = 2(2\pi)^{(n-1)/2} R^n \frac{\sqrt{\pi}}{2^{(n+1)/2}\Gamma(n+1/2)} = \frac{\pi^{n/2}}{\Gamma(n+1/2)} R^n
\end{aligned}$$

(Aus Bsp.1b $n!! = 2^{(n+1)/2}\Gamma(n+1/2)/\sqrt{\pi}$)
Alternative Lösung :

$$\begin{aligned}
V_n(R) &= \sqrt{\pi} R \frac{\Gamma((n+1)/2)}{\Gamma(n/2+1)} V_{n-1}(R) \\
&= \pi R^2 \frac{\Gamma((n+1)/2)\Gamma(n/2)}{\Gamma(n/2+1)\Gamma(n/2+1/2)} V_{n-2}(R) = \pi R^2 \frac{\Gamma(n/2)}{\Gamma(n/2+1)} V_{n-2}(R) \\
&= (\pi)^{3/2} R^3 \frac{\Gamma(n/2)\Gamma(n/2-1/2)}{\Gamma(n/2+1)\Gamma(n/2)} V_{n-3}(R) = (\pi)^{3/2} R^3 \frac{\Gamma(n/2-1/2)}{\Gamma(n/2+1)} V_{n-3}(R) \\
&\vdots \\
&= (\pi)^{(n-2)/2} R^{n-2} \frac{\Gamma(2)}{\Gamma(n/2+1)} \underbrace{V_2(R)}_{=\pi R^2} = (\pi)^{n/2} R^n \frac{1}{\Gamma(n/2+1)}
\end{aligned}$$

11.3 Fuchssche Klasse

$x(x-1)y'' + ((1+a+b)x-c)y' + aby = 0 \to a_2(x) = x(x-1)$, $a_1(x) = (1+a+b)x-c$, $a_0(x) = ab$.
Singulärer Punkt an $x_0 = 0$:
$\alpha_0 = \lim_{x\to 0} x\frac{a_1(x)}{a_2(x)} = \lim_{x\to 0} \frac{(1+a+b)x-c}{x-1} = c$, $\beta_0 = \lim_{x\to 0} x^2 \frac{a_0(x)}{a_2(x)} = \lim_{x\to 0} \frac{abx}{x-1} = 0$ (regulärer singulärer Punkt)
Charakteristische Exponenten:
$\sigma(\sigma-1) + \alpha_0\sigma + \beta_0 = \sigma^2 + (c-1)\sigma \to \sigma = 0, 1-c$.
Singulärer Punkt an $x_0 = 1$:
$\alpha_0 = \lim_{x\to 1}(x-1)\frac{a_1(x)}{a_2(x)} = \lim_{x\to 1} \frac{(1+a+b)x-c}{x} = 1+a+b-c$, $\beta_0 = \lim_{x\to 1}(x-1)^2 \frac{a_0(x)}{a_2(x)} = \lim_{x\to 1} \frac{ab(x-1)}{x} = 0$ (regulärer singulärer Punkt)
Charakteristische Exponenten:
$\sigma = 0, c-a-b$.
Singulärer Punkt an $x_0 = \infty$:
Transformation $x = 1/t \to u'' + \tilde{p}_1(t)u' + \tilde{p}_2(t)u = 0$ mit $\tilde{p}_1(t) = 2/t - (1/t^2)a_1(1/t)/a_2(1/t)$ und $\tilde{p}_2(t) = (1/t^4)a_0(1/t)/a_2(1/t)$
$\alpha_0 = \lim_{t\to 0} t\tilde{p}_1(t) = \lim_{t\to 0} \frac{(c-2)t+a+b-1}{t-1} = 1-a-b$
$\beta_0 = \lim_{t\to 0} t^2\tilde{p}_2(t) = \lim_{t\to 0} \frac{ab}{1-t} = ab$ (regulärer singulärer Punkt)
Charakteristische Exponenten:
$\sigma^2 - (a+b)\sigma + ab = 0 \to \sigma = a, b$.
Ansatz (Entwicklung am regulären singulären Punkt $x = 0$) : $y(x) = \sum_{\ell=0}^{\infty} w_\ell x^{\ell+\sigma}$
$y'(x) = \sum_{\ell=0}^{\infty}(\ell+\sigma)w_\ell x^{\ell+\sigma-1} = \sum_{\ell=-1}^{\infty}(\ell+\sigma+1)w_{\ell+1}x^{\ell+\sigma}$
$y''(x) = \sum_{\ell=0}^{\infty}(\ell+\sigma-1)(\ell+\sigma)w_\ell x^{\ell+\sigma-2} = \sum_{\ell=-1}^{\infty}(\ell+\sigma)(\ell+\sigma+1)w_{\ell+1}x^{\ell+\sigma-1}$

$$
\begin{aligned}
& x(x-1)y'' + ((1+a+b)x-c)y' + aby = \sum_{\ell=0}^{\infty}(\ell+\sigma-1)(\ell+\sigma)w_\ell x^{\ell+\sigma} - \sum_{\ell=-1}^{\infty}(\ell+\sigma)(\ell+\sigma+1)w_{\ell+1}x^{\ell+\sigma} \\
& +(1+a+b)\sum_{\ell=0}^{\infty}(\ell+\sigma)w_\ell x^{\ell+\sigma} - c\sum_{\ell=-1}^{\infty}(\ell+\sigma+1)w_{\ell+1}x^{\ell+\sigma} + ab\sum_{\ell=0}^{\infty}w_\ell x^{\ell+\sigma} \\
= \; & -\underbrace{(\sigma(\sigma-1)+c\sigma)}_{=0}w_0x^{\sigma-1} + \sum_{\ell=0}^{\infty}((\ell+\sigma-1)(\ell+\sigma)w_\ell - (\ell+\sigma)(\ell+\sigma+1)w_{\ell+1} \\
& +(1+a+b)(\ell+\sigma)w_\ell - c(\ell+\sigma+1)w_{\ell+1} + abw_\ell)x^{\ell+\sigma} \\
= \; & \sum_{\ell=0}^{\infty}(((\ell+\sigma+a+b)(\ell+\sigma)+ab)w_\ell - (\ell+\sigma+c)(\ell+\sigma+1)w_{\ell+1})x^{\ell+\sigma} = 0
\end{aligned}
$$

Rekursionsrelation :

$$
w_{\ell+1} = \frac{(\ell+\sigma+a+b)(\ell+\sigma)+ab}{(\ell+\sigma+c)(\ell+\sigma+1)}w_\ell = \frac{(\ell+\sigma+a)(\ell+\sigma+b)}{(\ell+\sigma+c)(\ell+\sigma+1)}w_\ell
$$

Lösung

$$
\begin{aligned}
w_\ell &= \frac{(\ell+\sigma+a-1)(\ell+\sigma+b-1)}{(\ell+\sigma+c-1)(\ell+\sigma)}w_{\ell-1} = \frac{(\ell+\sigma+a-1)(\ell+\sigma+a-2)(\ell+\sigma+b-1)(\ell+\sigma+b-2)}{(\ell+\sigma+c-1)(\ell+\sigma+c-2)(\ell+\sigma)(\ell+\sigma-1)}w_{\ell-2} \\
&= \cdots = \frac{\Gamma(\ell+\sigma+a)\Gamma(\ell+\sigma+b)}{\Gamma(\ell+\sigma+c)\Gamma(\ell+\sigma+1)}\frac{\Gamma(\sigma+c)\Gamma(\sigma+1)}{\Gamma(\sigma+a)\Gamma(\sigma+b)}w_0 = \frac{(\sigma+a)_\ell(\sigma+b)_\ell}{(\sigma+c)_\ell(\sigma+1)_\ell}w_0
\end{aligned}
$$

Wenn $\sigma = 0$, $y(x) = \sum_{\ell=0}^{\infty} w_\ell x^\ell$ mit

$$
w_\ell = \frac{(a)_\ell(b)_\ell}{(c)_\ell \; \ell!}w_0 \,.
$$

Diese Lösung $y(x)$ ist bekannt als Hypergeometrische Funktion.