1. Test - Lösungen **4.12.2015**

1 Spektraltheorem [30 Punkte]

a)[12]

$\mathbf{A} = \begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix}$

Für den Eigenvektor $\mathbf{x}$, gilt die Eigenwertgleichung $\mathbf{Ax} = \lambda\mathbf{x} \to \mathbf{x} \times \mathbf{Ax} = 0$

Wenn $\mathbf{x} = (x, y)$, $xRx - yRy = 0 \to Rx^2 - Ry^2 = 0 \to y = \pm x$.

Für den Eigenvektor $\mathbf{x}_1^T = (1, 1)$, $\mathbf{Ax}_1 = R\mathbf{x}_1$, d.h. der Eigenwert ist $\lambda_1 = R$.

Für den Eigenvektor $\mathbf{x}_2^T = (1, -1)$, $\mathbf{Ax}_2 = -R\mathbf{x}_2$, d.h. der Eigenwert ist $\lambda_2 = -R$.

(Allgemein sind die Eigenvektoren $\mathbf{x}_1^T = c_1(1, 1)$ und $\mathbf{x}_2^T = c_2(1, -1)$ mit beliebigen Konstanten c_1 und c_2 korrekt außer $c_1 = 0$ und $c_2 = 0$.)

$\mathbf{x}_1 \cdot \mathbf{x}_2 = 1 - 1 = 0 \to \{\mathbf{x}_1, \mathbf{x}_2\}$ ist eine orthogonale Basis.

Alternativ werden die Eigenwerten mit der Gleichung $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ gerechnet. $\to \lambda^2 - R^2 = 0 \to \lambda = \pm R$

b)[18]

Duale Basis: $\mathbf{x}^1 = c(1, 1) = (1/2)(1, 1)$ und $\mathbf{x}^2 = c(1, -1) = (1/2)(1, -1)$ (da $\mathbf{x}^j \cdot \mathbf{x}_j = 1$)

$\to \mathbf{x}^j\mathbf{A}\mathbf{x}_k = \lambda_j {\delta^j}_k \to \mathbf{X}^{-1}\mathbf{AX} = \mathrm{diag}(R, -R) \to \mathbf{A} = \mathbf{XDX}^{-1}$ wobei $\mathbf{X} = (\mathbf{x}_1\ \mathbf{x}_2)$ und $\mathbf{D} = \mathrm{diag}(R, -R)$.

(Statt der expliziten Rechnung der dualen Basis kann $\mathbf{X}^{-1}$ als die Inverse der Matrix $\mathbf{X}$ gerechnet werden)

$\to \exp(\mathbf{A}) = \mathbf{X} \begin{pmatrix} e^R & 0 \\ 0 & e^{-R} \end{pmatrix} \mathbf{X}^{-1}$

$\exp(\mathbf{A}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{X} \begin{pmatrix} e^R & 0 \\ 0 & e^{-R} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\mathbf{X} \begin{pmatrix} e^R \\ e^{-R} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^R + e^{-R} \\ e^R - e^{-R} \end{pmatrix} \left(\text{oder } = \begin{pmatrix} \cosh(R) \\ \sinh(R) \end{pmatrix}\right)$

Alternative Lösung:

Projektor : $\mathbf{E}_\mathbf{x} = \frac{1}{|\mathbf{x}|^2}\mathbf{x} \otimes \mathbf{x}^T \to \mathbf{E}_{\mathbf{x}_1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{E}_{\mathbf{x}_2} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$

$\mathbf{A} = R\,\mathbf{E}_{\mathbf{x}_1} - R\,\mathbf{E}_{\mathbf{x}_2} \to \exp(\mathbf{A}) = e^R\,\mathbf{E}_{\mathbf{x}_1} + e^{-R}\mathbf{E}_{\mathbf{x}_2}$

$\exp(\mathbf{A}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}e^R \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^R \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^R + e^{-R} \\ e^R - e^{-R} \end{pmatrix} = \begin{pmatrix} \cosh(R) \\ \sinh(R) \end{pmatrix}$

Alternative Lösung 2:

$\mathbf{A}^2 = \begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix} \begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix} = R^2\mathbf{I}\ \mathbf{A}^3 = R^2\mathbf{A} = R^3\tilde{\mathbf{A}}$ wobei $\tilde{\mathbf{A}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Für gerades $n = 2m$, $\mathbf{A}^{2m} = R^{2m}\mathbf{I}$ und, für ungerades $n = 2m + 1$, $\mathbf{A}^{2m+1} = R^{2m+1}\tilde{\mathbf{A}}$

$e^\mathbf{A} = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n = \sum_{m=0}^{\infty} \frac{1}{(2m)!}R^{2m}\mathbf{I} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}R^{2m+1}\tilde{\mathbf{A}}$

$= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{1}{n!}R^n + \sum_{n=0}^{\infty} \frac{1}{n!}(-R)^n\right)\mathbf{I} + \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{1}{n!}R^n - \sum_{n=0}^{\infty} \frac{1}{n!}(-R)^n\right)\tilde{\mathbf{A}}$

$= \frac{1}{2}(e^R + e^{-R})\mathbf{I} + \frac{1}{2}(e^R - e^{-R})\tilde{\mathbf{A}}$

$e^\mathbf{A} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^R + e^{-R} \\ e^R - e^{-R} \end{pmatrix}$

2 Lokale Transformation [40 Punkte]

a)[12]

Kartesische Basis: $\mathbf{e}_1 = \hat{\mathbf{x}}$, $\mathbf{e}_2 = \hat{\mathbf{y}}$

Polarkoordinaten: $(x'^1, x'^2) = (r, \theta)$

Transformation der Koordinaten: $x^1 = r\cos\theta$, $x^2 = r\sin\theta$.

Transformation der Basis: $\mathbf{dx} = dx^i\mathbf{e}_i = dx'^j(\partial'_j x^i)\mathbf{e}_i = dx'^j\mathbf{e}'_j \to \mathbf{e}'_j = \partial'_j x^i\mathbf{e}_i \quad (\partial'_j x^i = \frac{\partial x^i}{\partial x'^j})$

$$\left.\begin{aligned} \mathbf{e}'_1 &= \frac{\partial x^1}{\partial x'^1}\mathbf{e}_1 + \frac{\partial x^2}{\partial x'^1}\mathbf{e}_2 = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2 \\ \mathbf{e}'_2 &= \frac{\partial x^1}{\partial x'^2}\mathbf{e}_1 + \frac{\partial x^2}{\partial x'^2}\mathbf{e}_2 = -r\sin\theta\,\mathbf{e}_1 + r\cos\theta\,\mathbf{e}_2 \end{aligned}\right\} \to (\mathbf{e}'_1\ \mathbf{e}'_2) = (\mathbf{e}_1\ \mathbf{e}_2)\,\mathbf{T} \text{ mit } \mathbf{T} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Alternative Lösung ohne Indexschreibweise

$\mathbf{dx} \to \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta \\ \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial r} \end{pmatrix} dr + \begin{pmatrix} \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \theta} \end{pmatrix} d\theta \to (\mathbf{e}'_1\,\mathbf{e}'_2) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}$

$\mathbf{T} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

b) [12]

$g_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j = a^k{}_i \mathbf{e}_k a^l{}_j \mathbf{e}_l = a^k{}_i a^l{}_j \delta_{kl} = a^k{}_i a^k{}_j \to \mathbf{g} = \mathbf{T}^T\mathbf{T} \to \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ und $\mathbf{g}^{-1} \to \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$

Alternative Lösung:

Duale Basis: $\mathbf{e}^1 = \mathbf{e}_1^T$, $\mathbf{e}^2 = \frac{1}{r^2}\mathbf{e}_2^T$.

$g_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j \to \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, $g^{ij} = \mathbf{e}'^i \cdot \mathbf{e}'^j \to \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$

c) [16]

$ds^2 = dx^i dx_i = dx'^i dx'_i = g_{ij} dx'^i dx'^j = dr^2 + r^2 d\theta^2$

Parametrisierung: $r = a\sin^2(\pi t/T)$, $\theta = 2\pi t/T \to dr = 2\pi\frac{a}{T}\sin(\pi t/T)\cos(\pi t/T)dt$, $d\theta = 2(\pi/T)dt$.

$\sqrt{ds^2} = \left|\frac{d}{dt}\vec{x}(t)\right| dt = \left|\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right| dt = \sqrt{4\pi^2\frac{a^2}{T^2}\sin^2(\pi t/T)\cos^2(\pi t/T) + a^2\sin^4(\pi t/T)4\pi^2/T^2 dt}$

$= 2\pi\frac{a}{T}|\sin(\pi t/T)|\sqrt{\cos^2(\pi t/T) + \sin^2(\pi t/T)}dt = 2\pi\frac{a}{T}|\sin(\pi t/T)|dt$

$\to \int_C ds = \int_0^T 2\pi\frac{a}{T}|\sin(\pi t/T)|dt = -2\pi\frac{a}{T}\frac{T}{\pi}\cos(\pi t/T)\Big|_0^T = 4a$

3 Multiple Choice Fragen [30 Punkte, 3 Punkte je Frage]

1) $\delta^i{}_i\delta^j{}_j + \delta^i{}_j\delta^j{}_i = 3\times 3 + \delta^i_i = 9 + 3 = 12$

2) $g^{ij}g_{ji} = \delta^i_i = 3$

3) $\mathrm{grad}|\mathbf{x}|^2 \to \partial_i x_j x_j = \delta_{ij}x_j + x_j\delta_{ij} = 2x_i \to 2\mathbf{x}$

4) $\mathrm{grad}|\mathbf{x}| \to \partial_i\sqrt{x_jx_j} = \frac{1}{2}(x_jx_j)^{-1/2}(\delta_{ij}x_j + x_j\delta_{ij}) = \frac{1}{2}(x_jx_j)^{-1/2}2x_i = x_i/|\mathbf{x}| \to \mathbf{x}/|\mathbf{x}|$

5) $[\mathbf{A},\mathbf{BC}] = \mathbf{ABC} - \mathbf{BCA} = \mathbf{ABC} - \mathbf{BAC} + \mathbf{BAC} - \mathbf{BCA} = [\mathbf{A},\mathbf{B}]\mathbf{C} + \mathbf{B}[\mathbf{A},\mathbf{C}]$

6) $\int_F \mathrm{rot}\,\mathbf{b}\cdot d\mathbf{A} = \int_F \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dA = \int_F dA = \pi a^2$

7) parametrisierung: $(x,y,z) = (0, a\cos t, a\sin t)$

$\oint_D \mathbf{b}\cdot d\mathbf{s} = \int_0^{2\pi} \begin{pmatrix} 0 \\ a\sin t \\ a\cos t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -a\sin t \\ a\cos t \end{pmatrix} dt = a^2\int_0^{2\pi}(-\sin^2 t + \cos^2 t)dt = a^2\int_0^{2\pi}\cos(2t)dt = 0$

8) $\oint_C \frac{z}{2z-1}dz = 2\pi i\frac{1}{4} = \frac{1}{2}\pi i$

9) $\oint_C \frac{z+1}{z(z+3)}dz = \frac{2}{3}\pi i$

10) γ: geschlossener unterer Halbkreis, γ_o: offener unterer Halbkreis

$\int_{-\infty}^{\infty} \frac{z+1}{(z-i)(z+1-i)(z+1+i)}dz = -\oint_\gamma \frac{z+1}{(z-i)(z+1-i)(z+1+i)}dz + \int_{\gamma_o} \frac{z+1}{(z-i)(z+1-i)(z+1+i)}dz$

$= -\oint_\gamma \frac{z+1}{(z-i)(z+1-i)(z+1+i)}dz + \underbrace{\int_{-\pi}^{0} \frac{Re^{i\theta}+1}{R^3e^{3i\theta}+\dots}iRe^{i\theta}d\theta}_{=0 \text{ wenn } R\to\infty}$

$= -(2\pi i)\frac{-i}{(-1-2i)(-2i)} = -2\pi\frac{1}{2i-4} = 2\pi\frac{2i+4}{20} = \pi\frac{2+i}{5}$