**2. Test - Lösungen** **22.1.2016**

## 1 Differentialgleichung [30 Punkte]

a)[15]
Differentailgleichung: $\left(\partial_u^2 + 2u^{-1}\partial_u + u^{-2}(1-v^2)\partial_v^2 - 2u^{-2}v\partial_v\right)\psi(u,v) = -2E\psi(u,v)$
Ansatz: $\psi(u,v) = F(u)G(v)$
$G\partial_u^2 F + 2u^{-1}G\partial_u F + u^{-2}(1-v^2)F\partial_v^2 G - 2u^{-2}vF\partial_v G = -2EFG$
$\to u^2F^{-1}\partial_u^2F + 2uF^{-1}\partial_uF + (1-v^2)G^{-1}\partial_v^2G - 2vG^{-1}\partial_vG = -2u^2E$
$\to u^2F^{-1}\partial_u^2F + 2uF^{-1}\partial_uF + 2u^2E = -(1-v^2)G^{-1}\partial_v^2G + 2vG^{-1}\partial_vG = C$
getrennte Gleichungen
$u^2\partial_u^2F + 2u\partial_uF + 2u^2EF = CF$
und
$(1-v^2)\partial_v^2G - 2v\partial_vG = -CG$
b)[15]
Sturm-Liouville'sche Gestalt $\partial_u p(u)\partial_u F + q(u)F = 0 \to p(u)F'' + p'(u)F'(u) + q(u)F = 0$
Vergleich der Koeffizienten
$\frac{p'(u)}{p(u)} = \frac{2}{u} \to \ln p(u) = 2\ln u + A_1 \to p(u) = A_2u^2$
$\frac{q(u)}{p(u)} = \frac{2u^2E-C}{u^2} \to q(u) = p(u)(2E - Cu^{-2}) = A_2(2Eu^2 - C)$
z.B. wenn $A_2 = 1$,
$\partial_u u^2\partial_u F + (2Eu^2 - C)F = 0$

## 2 Greensche Fnktion [40 Punkte]

a)[24]
$\mathcal{L}_t y(t) = \left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right)y(t) = f(t)$
Inhomogene Gleichung: $\mathcal{L}_t G_I(t,t') = \delta(t-t')$
Mit dem Einsetzen des Ansatzes $G_I(t,t') = (2\pi)^{-1}\int_{-\infty}^{\infty}\tilde{G}_I(\omega)\,e^{i\omega(t-t')}d\omega$
$\to (2\pi)^{-1}\int_{-\infty}^{\infty}(-\omega^2 + 3i\omega + 2)\tilde{G}_I(\omega)\,e^{i\omega(t-t')}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega(t-t')}d\omega$
Vergleich der Integranden $\tilde{G}_I(\omega) = -\frac{1}{\omega^2-3i\omega-2} = -\frac{1}{(\omega-2i)(\omega-i)}$
Fourier-Transformation $G_I(t,t') = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega(t-t')}}{(\omega-2i)(\omega-i)}d\omega$
Zwei Pole liegen in der oberen Halbebene der komplexen Zahlen und kein Pol liegt in der unteren Halbebene. Mit Hilfe des Residuensatzes (Integrationspfad : oberer Halbkreis $C_1$ für $t > t'$ und unterer Halbkreis $C_2$ für $t < t'$) ergibt sich
$G_I(t,t') = \frac{1}{2\pi}H(t-t')\left(\oint_{C_1}\tilde{G}_I(\omega)e^{i\omega(t-t')}d\omega - \int_{C_1}\tilde{G}_I(\omega)e^{i\omega(t-t')}d\omega\right) = -\frac{1}{2\pi}H(t-t')\oint_{C_1}\frac{e^{i\omega(t-t')}}{(\omega-2i)(\omega-i)}d\omega$
$= -iH(t-t')\left[\mathrm{Res}_{\omega\to 2i}\frac{e^{i\omega(t-t')}}{(\omega-2i)(\omega-i)} + \mathrm{Res}_{\omega\to i}\frac{e^{i\omega(t-t')}}{(\omega-2i)(\omega-i)}\right] = H(t-t')\left(-e^{-2(t-t')} + e^{-(t-t')}\right)$
b) [16]
$y_I(t) = \int_{-\infty}^{\infty}G_I(t,t')f(t')dt' = \int_{-\infty}^{\infty}H(t-t')\left(-e^{-2(t-t')} + e^{-(t-t')}\right)AH(t')dt'$
Wenn $t < 0$, $y_I(t) = 0$.
Wenn $t > 0$, $y_I(t) = A\int_0^t\left(-e^{-2(t-t')} + e^{-(t-t')}\right)dt' = -Ae^{-2t}\,\frac{1}{2}e^{2t'}\Big|_{t'=0}^{t} + Ae^{-t}\,e^{t'}\Big|_{t'=0}^{t}$
$= -\frac{A}{2}e^{-2t}(e^{2t}-1) + e^{-t}(e^t - 1) = \frac{A}{2}(1 + e^{-2t} - 2e^{-t})$
$\to y_I(0) = 0$
Wenn $t < 0$, $y_I'(t) = 0$
Wenn $t > 0$, $y_I'(t) = -Ae^{-2t} + Ae^t$
$\to y_I'(0) = 0$

## 3 Multiple Choice Fragen [30 Punkte, 3 Punkte je Frage]

1) $\int_{-\infty}^{\infty}\delta(2x)(x^2+2)dx = \int_{-\infty}^{\infty}\delta(t)\left(\frac{1}{4}t^2 + 2\right)\frac{1}{2}dt = 1$
2) $t = 4x^2$. For $x < 0$, $x = -\sqrt{t}/2 \to dx = -1/(4\sqrt{t})dt$
$\int_{-\infty}^{0}\delta(4x^2-1)(10x+1)dx = \int_{\infty}^{0}\delta(t-1)\left(-5\sqrt{t}+1\right)\left(-\frac{1}{4\sqrt{t}}\right)dt = \int_0^{\infty}\delta(t-1)\left(-5\sqrt{t}+1\right)\frac{1}{4\sqrt{t}}dt = -1$

3) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|1-xy|\delta(2x-y)\delta(x-1)dxdy=\int_{-\infty}^{\infty}|1-y|\delta(2-y)dy=1$
4) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(E-x^2-y^2)dxdy=\pi E$ (Fläche des Kreises mit Radius $\sqrt{E}$)
5) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(E-x^2-y^2)dxdy=\frac{d}{dE}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(E-x^2-y^2)dxdy=\frac{d}{dE}(\pi E)=\pi$
6) $\frac{d}{dx}\left(H(x)e^{-x}\sin x\right)=\delta(x)e^{-x}\sin x-H(x)e^{-x}\sin x+H(x)e^{-x}\cos x=H(x)e^{-x}(\cos x-\sin x)$
7) $\frac{d}{dx}\sin|x|=\frac{d}{dx}(H(x)\sin x-H(-x)\sin x)=\delta(x)\sin x+H(x)\cos x+\delta(-x)\sin x-H(-x)\cos x)=(H(x)-H(-x))\cos x$
8) $\int_0^{\infty}f'(x)\sin xdx=f(x)\sin x|_{x=0}^{\infty}-\int_0^{\infty}f(x)\cos xdx=-\int_0^{\pi/2}\sin x\cos xdx=-\frac{1}{2}\int_0^{\pi/2}\sin 2xdx=-\frac{1}{2}$
Alternative Lösung
$f(x)=H(\pi/2+x)H(\pi/2-x)\sin x$
$\rightarrow f'(x)=H(\pi/2+x)H(\pi/2-x)\cos x+\delta(\pi/2+x)H(\pi/2-x)\sin x-H(\pi/2+x)\delta(\pi/2-x)\sin x$
$=H(\pi/2+x)H(\pi/2-x)\cos x-\delta(\pi/2+x)H(\pi/2-x)-H(\pi/2+x)\delta(\pi/2-x)$
$\int_0^{\infty}f'(x)\sin xdx$
$=\int_0^{\infty}H(\pi/2+x)H(\pi/2-x)\cos x\sin xdx-\int_0^{\infty}\delta(\pi/2+x)H(\pi/2-x)\sin xdx-\int_0^{\infty}H(\pi/2+x)\delta(\pi/2-x)\sin xdx$
$=\int_0^{\pi/2}\cos x\sin xdx-0-\int_0^{\infty}\delta(\pi/2-x)\sin xdx=-1/2$
9) $\Gamma(5)=4\Gamma(4)=12\Gamma(3)=24\Gamma(2)=24$
10) $t=x^2$. Wenn $x>0$, $x=\sqrt{t}$ und $dx=1/(2\sqrt{t})dt$
$\int_{-\infty}^{\infty}x^4e^{-x^2}dx=2\int_0^{\infty}x^4e^{-x^2}dx=2\int_0^{\infty}t^2e^{-t}\frac{1}{2\sqrt{t}}dt=\int_0^{\infty}t^{3/2}e^{-t}dt=\Gamma(5/2)=(3/2)(1/2)\Gamma(1/2)=3\sqrt{\pi}/4$