10. Tutorium - Lösungen **15.1.2016**

- ANMERKUNG: Es liegt in der Verantwortung des Einzelnen, sich die Beispiele zunächst alleine und ganz ohne Hilfsmittel zu rechnen. Google, Wolfram Alpha, Lösungssammlungen, etc. helfen nur kurzfristig - leider nicht beim Test!

10.1 Multiple Choice Fragen

a) $\left|\left(ix-\frac{1}{2}\right)!\right|^2 = \left|\Gamma\left(ix+\frac{1}{2}\right)\right|^2 = \Gamma\left(ix+\frac{1}{2}\right)\Gamma\left(-ix+\frac{1}{2}\right) = \Gamma\left(ix+\frac{1}{2}\right)\Gamma\left(1-\left(ix+\frac{1}{2}\right)\right) = \frac{\pi}{\sin(\pi(ix+1/2))}$
$= \frac{2\pi i}{e^{i\pi(ix+1/2)}-e^{-i\pi(ix+1/2)}} = \frac{2\pi i}{ie^{-\pi x}+ie^{\pi x}} = \frac{\pi}{\cosh(\pi x)}$
b) $\left|e^{-\pi\gamma/2}\Gamma\left(1+i\gamma\right)\right|^2 = e^{-\pi\gamma}\Gamma\left(1+i\gamma\right)\Gamma\left(1-i\gamma\right) = e^{-\pi\gamma}(i\gamma)\Gamma\left(i\gamma\right)\Gamma\left(1-i\gamma\right) = e^{-\pi\gamma}(i\gamma)\frac{\pi}{\sin(\pi i\gamma)}$
$= e^{-\pi\gamma}(i\gamma)\frac{2\pi i}{e^{i\pi i\gamma}-e^{-i\pi i\gamma}} = -e^{-\pi\gamma}\gamma\frac{2\pi}{e^{-\pi\gamma}-e^{\pi\gamma}} = \frac{2\pi\gamma}{e^{2\pi\gamma}-1}$
c) $\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right) = \left[\left(\frac{1}{2}-n\right)^{-1}\Gamma\left(\frac{1}{2}-n+1\right)\right]\left[\left(\frac{1}{2}+n-1\right)\Gamma\left(\frac{1}{2}+n-1\right)\right] = -\Gamma\left(\frac{1}{2}-n+1\right)\Gamma\left(\frac{1}{2}+n-1\right)$
$= (-1)^2\Gamma\left(\frac{1}{2}-n+2\right)\Gamma\left(\frac{1}{2}+n-2\right) = \cdots = (-1)^n\Gamma\left(\frac{1}{2}-n+n\right)\Gamma\left(\frac{1}{2}+n-n\right) = (-1)^n\pi$
d) $t=x^2 \to x=\pm\sqrt{t} \to dx = \pm\frac{1}{2t^{1/2}}dt$
$\int_{-\infty}^{\infty} x^{2a}e^{-x^2}dx = \int_0^{\infty} x^{2a}e^{-x^2}dx + \int_{-\infty}^{0} x^{2a}e^{-x^2}dx = \int_0^{\infty}(\sqrt{t})^{2a}e^{-t}\frac{1}{2t^{1/2}}dt + \int_{\infty}^{0}(-\sqrt{t})^{2a}e^{-t}\left(-\frac{1}{2t^{1/2}}\right)dt$
$= \int_0^{\infty} t^a e^{-t}\frac{1}{2t^{1/2}}dt + (-1)^{2a}\int_0^{\infty} t^a e^{-t}\frac{1}{2t^{1/2}}dt = \left(1+(-1)^{2a}\right)\int_0^{\infty} t^a e^{-t}\frac{1}{2t^{1/2}}dt$
$= \frac{1}{2}\left(1+(-1)^{2a}\right)\int_0^{\infty} t^{a-1/2}e^{-t}dt = \frac{1}{2}\left(1+(-1)^{2a}\right)\int_0^{\infty} t^{(a+1/2)-1}e^{-t}dt = \frac{1}{2}\left(1+(-1)^{2a}\right)\Gamma\left(a+\frac{1}{2}\right)$
e) $s=-\ln x \to x=e^{-s}\ dx = -e^{-s}ds$
$-\int_0^1 x^a\ln x\,dx = \int_0^{\infty} e^{-(a+1)s}s\,ds$
$t=(a+1)s \to ds=(a+1)^{-1}dt$
Wenn $a+1>0$, Integrationsbereich $0<s<\infty \to 0<t<\infty$
$\int_0^{\infty} e^{-(a+1)s}s\,ds = \int_0^{\infty} e^{-t}\frac{t}{(a+1)^2}\,dt = \frac{1}{(a+1)^2}\int_0^{\infty} t\,e^{-t}\,dt = \frac{1}{(a+1)^2}\Gamma(2) = \frac{1}{(a+1)^2}$
Wenn $a+1<0$, Integrationsbereich $0<s<\infty \to 0>t>-\infty$
$\int_0^{\infty} e^{-(a+1)s}s\,ds = \int_0^{-\infty} e^{-t}\frac{t}{(a+1)^2}\,dt = -\frac{1}{(a+1)^2}\int_{-\infty}^{0} t\,e^{-t}\,dt = \infty$
In diesen Fall gibt es 2 richtige Lösungen in den Multiple-Choice.
f) $t=x^4 \to dt = 4x^3dx = 4t^{3/4}dx$
$\int_0^{\infty} e^{-x^4}dx = \int_0^{\infty} e^{-t}\frac{1}{4}t^{-3/4}dt = \frac{1}{4}\Gamma\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)!$

10.2 Greensche Funktion

a)
$\tilde{G}_I(\omega) = -\frac{1}{(\omega-E_1)(\omega-E_2)}$
$G_I^+(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(\omega)e^{i\omega(t-t')}d\omega = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(\omega-E_1)(\omega-E_2)}e^{i\omega(t-t')}d\omega$
Das Integral wird mit einer Verschiebung der Pole bei $i\varepsilon$ ($\varepsilon>0$) berechnet.
$G_I^+(t,t') = -\frac{1}{2\pi}\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\frac{1}{(\omega-E_1-i\varepsilon)(\omega-E_2-i\varepsilon)}e^{i\omega(t-t')}d\omega$
$= -iH(t-t')\lim_{\varepsilon\to 0^+}\left(\frac{1}{E_1-E_2}e^{i(E_1+i\varepsilon)(t-t')} + \frac{1}{E_2-E_1}e^{i(E_2+i\varepsilon)(t-t')}\right)$
$= -iH(t-t')\frac{1}{E_1-E_2}\left(e^{iE_1(t-t')}-e^{iE_2(t-t')}\right) = -iH(t-t')\frac{1}{E_1-E_2}e^{i(E_1+E_2)(t-t')/2}\left(e^{i(E_1-E_2)(t-t')/2}-e^{-i(E_1-E_2)(t-t')/2}\right)$
$= 2H(t-t')\frac{1}{E_1-E_2}e^{i(E_1+E_2)(t-t')/2}\sin\left(\frac{1}{2}(E_1-E_2)(t-t')\right)$
$G_I^+(0,t'>0) = 2H(-t')\frac{1}{E_1-E_2}e^{-i(E_1+E_2)t'/2}\sin\left(-\frac{1}{2}(E_1-E_2)t'\right) = 0$
$G_I^{+\prime}(t,t') = 2\delta(t-t')\frac{1}{E_1-E_2}e^{i(E_1+E_2)(t-t')/2}\sin\left(\frac{1}{2}(E_1-E_2)(t-t')\right)$
$+2H(t-t')\partial_t\left(\frac{1}{E_1-E_2}e^{i(E_1+E_2)(t-t')/2}\sin\left(\frac{1}{2}(E_1-E_2)(t-t')\right)\right)$
$G_I^{+\prime}(0,t') = 2H(-t')\,\partial_t\left(\frac{1}{E_1-E_2}e^{i(E_1+E_2)(t-t')/2}\sin\left(\frac{1}{2}(E_1-E_2)(t-t')\right)\right)\Big|_{t=0} = 0$
b)
$G_I^-(t,t') = -\frac{1}{2\pi}\lim_{\varepsilon\to 0^-}\int_{-\infty}^{\infty}\frac{1}{(\omega-E_1-i\varepsilon)(\omega-E_2-i\varepsilon)}e^{i\omega(t-t')}d\omega$

$= iH(t'-t)\lim_{\varepsilon\to 0^-}\left(\frac{1}{E_1-E_2}e^{i(E_1+i\varepsilon)(t-t')}+\frac{1}{E_2-E_1}e^{i(E_2+i\varepsilon)(t-t')}\right)$
$= iH(t'-t)\frac{1}{E_1-E_2}\left(e^{iE_1(t-t')}-e^{iE_2(t-t')}\right) = iH(t'-t)\frac{1}{E_1-E_2}e^{i(E_1+E_2)(t-t')/2}\left(e^{i(E_1-E_2)(t-t')/2}-e^{-i(E_1-E_2)(t-t')/2}\right)$
$= -2H(t'-t)\frac{1}{E_1-E_2}e^{i(E_1+E_2)(t-t')/2}\sin\left(\frac{1}{2}(E_1-E_2)(t-t')\right)$
$G_I^-(0,t'>0) = -2H(t')\frac{1}{E_1-E_2}e^{-i(E_1+E_2)t'/2}\sin\left(-\frac{1}{2}(E_1-E_2)t'\right)\neq 0$
homogene Greensche Funktion $G_0(t,t') = 2\frac{1}{E_1-E_2}e^{-i(E_1+E_2)t'/2}\sin\left(-\frac{1}{2}(E_1-E_2)t'\right)$
$G(t,t') = G_I^-(t,t'>0)+G_0(t,t') = 2(1-H(t'-t))\frac{1}{E_1-E_2}e^{-i(E_1+E_2)t'/2}\sin\left(-\frac{1}{2}(E_1-E_2)t'\right)$
$= 2H(t-t')\frac{1}{E_1-E_2}e^{-i(E_1+E_2)t'/2}\sin\left(-\frac{1}{2}(E_1-E_2)t'\right) = G_I^+(t,t'>0)$

10.3 Separationsansatz der Kugelkoordinaten

a) $\nabla x^i = \mathbf{e}^k\partial_k x^i = g^{kj}\mathbf{e}_j\partial_k x^i = g^{kj}\mathbf{e}_j\delta^i{}_k = g^{ij}\mathbf{e}_j = \underbrace{g^{ii}\mathbf{e}_i = \frac{1}{g_{ii}}\mathbf{e}_i}_{\text{ohne Summe über } i}$ (g^{ij} ist diagonal für orthogonale Basis)

b) $g_{ii} = \mathbf{e}_i\cdot\mathbf{e}_i \to |\mathbf{e}_i| = \sqrt{g_{ii}}$.
$\mathbf{e}_1 = a\,\mathbf{e}_2\times\mathbf{e}_3 \to |\mathbf{e}_1| = a|\mathbf{e}_2\times\mathbf{e}_3| \to \sqrt{g_{11}} = a\sqrt{g_{22}}\sqrt{g_{33}} \to a = \sqrt{g_{11}/(g_{22}\,g_{33})}$
$\nabla\cdot(\frac{1}{G}\mathbf{e}_1) = \nabla\cdot(\frac{1}{G}a\,\mathbf{e}_2\times\mathbf{e}_3) = \nabla\cdot(\frac{1}{g_{22}}\mathbf{e}_2\times\frac{1}{g_{33}}\mathbf{e}_3) = (\nabla\times\underbrace{\frac{1}{g_{22}}\mathbf{e}_2}_{=\nabla x^2\text{ (Bsp.a)}})\cdot\frac{1}{g_{33}}\mathbf{e}_3 - \frac{1}{g_{22}}\mathbf{e}_2\cdot(\nabla\times\underbrace{\frac{1}{g_{33}}\mathbf{e}_3}_{=\nabla x^3})$

$= (\underbrace{\nabla\times\nabla x^2}_{=0})\cdot\frac{1}{g_{33}}\mathbf{e}_3 - \frac{1}{g_{22}}\mathbf{e}_2\cdot(\underbrace{\nabla\times\nabla x^3}_{=0}) = 0$
In ähnlicher Weise, $\nabla\cdot\left(\frac{1}{G}\mathbf{e}_2\right) = \nabla\cdot\left(\frac{1}{G}\mathbf{e}_3\right) = 0$
c) $\mathbf{v} = v^i\mathbf{e}_i$
$\nabla\cdot\mathbf{v} = \mathbf{e}^j\cdot\partial_j\left(v^i\mathbf{e}_i\right) = \mathbf{e}^j\cdot\partial_j\left(Gv^i\frac{1}{G}\mathbf{e}_i\right) = \mathbf{e}^j\cdot\partial_j\left(Gv^i\right)\frac{1}{G}\mathbf{e}_i + Gv^i\underbrace{\mathbf{e}^j\cdot\partial_j\left(\frac{1}{G}\mathbf{e}_i\right)}_{=0\text{(Bsp.b)}} = \frac{1}{G}\partial_j\left(Gv^i\right)\delta^j{}_i = \frac{1}{G}\partial_i\left(Gv^i\right)$

d) $\mathbf{v} = \nabla\psi(\mathbf{x}) = \mathbf{e}^j\partial_j\psi(\mathbf{x}) = g^{ij}\mathbf{e}_i\partial_j\psi(\mathbf{x})$
$\nabla^2\psi(\mathbf{x}) = \frac{1}{G}\partial_j\left(Gg^{ij}\partial_j\psi(\mathbf{x})\right) = \frac{1}{G}\partial_1\left(Gg^{11}\partial_1\psi(\mathbf{x})\right) + \frac{1}{G}\partial_2\left(Gg^{22}\partial_2\psi(\mathbf{x})\right) + \frac{1}{G}\partial_3\left(Gg^{33}\partial_3\psi(\mathbf{x})\right)$
$= \frac{1}{\sqrt{g_{11}\,g_{22}\,g_{33}}}\left(\partial_1\left(\sqrt{g_{22}\,g_{33}/g_{11}}\partial_1\psi(\mathbf{x})\right) + \partial_2\left(\sqrt{g_{11}\,g_{33}/g_{22}}\partial_2\psi(\mathbf{x})\right) + \partial_3\left(\sqrt{g_{11}\,g_{22}/g_{33}}\partial_3\psi(\mathbf{x})\right)\right)$
Für die Kugelkoordinaten, $g_{11} = 1$, $g_{22} = r^2$ und $g_{33} = r^2\sin^2\theta$
$\nabla^2\psi(\mathbf{x}) = \frac{1}{r^2\sin\theta}\left(\partial_r\left(r^2\sin\theta\partial_r\psi(\mathbf{x})\right) + \partial_\theta\left(\sin\theta\partial_\theta\psi(\mathbf{x})\right) + \partial_\phi\left((\sin\theta)^{-1}\partial_\phi\psi(\mathbf{x})\right)\right)$
$= \frac{1}{r^2}\partial_r\left(r^2\partial_r\psi(\mathbf{x})\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\psi(\mathbf{x})\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\psi(\mathbf{x})$
e) Ansatz: $\psi(\mathbf{x}) = R(r)P(\theta)F(\phi)$
$\mathcal{L}_r RPF + r^{-2}\mathcal{L}_\theta RPF + \frac{1}{r^2\sin^2\theta}\mathcal{L}_\phi RPF = 0$
dividieren mit RPF
$R^{-1}\mathcal{L}_r R + r^{-2}P^{-1}\mathcal{L}_\theta P + \frac{1}{r^2\sin^2\theta}F^{-1}\mathcal{L}_\phi F = 0$
multiplizieren mit r^2
$r^2R^{-1}\mathcal{L}_r R = -P^{-1}\mathcal{L}_\theta P - \frac{1}{\sin^2\theta}F^{-1}\mathcal{L}_\phi F$
linke Seite: Funktion von r, rechte Seite: Funktion von $\theta,\phi \to$ Die beide Seiten müssen konstant (unabhängig von r,θ,ϕ) sein.
$r^2R^{-1}\mathcal{L}_r R = Z_1 \to \mathcal{L}_r R - Z_1/r^2 R = 0$
$P^{-1}\mathcal{L}_\theta P + \frac{1}{\sin^2\theta}F^{-1}\mathcal{L}_\phi F = -Z_1$
multiplizieren mit $\sin^2\theta$
$\sin^2\theta P^{-1}\mathcal{L}_\theta P + Z_1\sin^2\theta = -F^{-1}\mathcal{L}_\phi F = Z_2$
$\to \mathcal{L}_\theta P + Z_1 P - \frac{Z_2}{\sin^2\theta}P = 0$
und
$\mathcal{L}_\phi F = -Z_2 F$
Die getrennte Differentialgleichungen sind
$\mathcal{L}_r R - Z_1/r^2 R = \frac{1}{r^2}\partial_r\left(r^2\partial_r R\right) - Z_1/r^2 R = 0$
$\mathcal{L}_\theta P + Z_1 P - \frac{Z_2}{\sin^2\theta}P = \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta P\right) + Z_1 P - \frac{Z_2}{\sin^2\theta}P = 0$
$\mathcal{L}_\phi F + Z_2 F = \partial_\phi^2 F + Z_2 F = 0$