7. Tutorium - Lösungen **11.12.2015**

- ANMERKUNG: Es liegt in der Verantwortung des Einzelnen, sich die Beispiele zunächst alleine und ganz ohne Hilfsmittel zu rechnen. Google, Wolfram Alpha, Lösungssammlungen, etc. helfen nur kurzfristig - leider nicht beim Test!

7.1 Multiple Choice Fragen

a) $$\underbrace{\int_{-\infty}^{\infty}\delta(2x)\frac{1}{\sqrt{4+x^2}}dx}_{y=2x\to dx=(1/2)dy} = \int_{-\infty}^{\infty}\delta(y)\frac{1}{\sqrt{4+y^2/4}}\frac{1}{2}dy = \frac{1}{\sqrt{4}}\frac{1}{2} = \frac{1}{4}$$

b) $$\underbrace{\int_{-\infty}^{\infty}\delta(x^2-1)e^x dx}_{y=x^2-1\to dy=2xdx} = \underbrace{\int_{\infty}^{-1}\delta(y)e^{x(y)}\frac{1}{2x(y)}dy}_{\text{Wenn} x\to-\infty,\ y\to\infty \text{ und wenn } x\to 0,\ y\to -1} + \underbrace{\int_{-1}^{\infty}\delta(y)e^{x(y)}\frac{1}{2x(y)}dy}_{\text{Wenn } x\to 0,\ y\to -1 \text{ und wenn} x\to\infty,\ y\to\infty}$$

$$= -\underbrace{\int_{-1}^{\infty}\delta(y)e^{x(y)}\frac{1}{2x(y)}dy}_{\text{Für } -\infty<x<0, y=0 \text{wenn } x=-1} + \underbrace{\int_{-1}^{\infty}\delta(y)e^{x(y)}\frac{1}{2x(y)}dy}_{\text{Für } 0<x<\infty, y=0\text{wenn } x=1} = -e^{-1}\frac{1}{-2} + e^{1}\frac{1}{2} = \frac{1}{2}(e+e^{-1})$$

c)
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|xy-1|\delta(x-2y)\delta(x^2-1)dxdy = \int_{-\infty}^{\infty}\frac{1}{2}|x\tfrac{1}{2}x-1|\delta(x^2-1)dx = \frac{1}{4}\int_{-\infty}^{\infty}|x^2-2|\delta(x^2-1)dx$$
$$= \frac{1}{4}\left(\left.\frac{|x^2-2|}{|2x|}\right|_{x=-1} + \left.\frac{|x^2-2|}{|2x|}\right|_{x=1}\right) = \frac{1}{4}$$

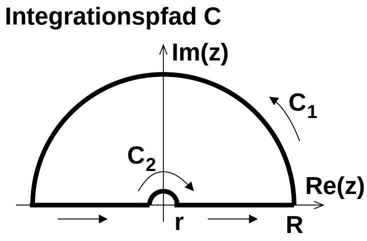
d)
$$\underbrace{\int_{-\infty}^{\infty}H(4-x^2)dx}_{y=4-x^2\to dy=-2xdx} = \underbrace{\int_{-\infty}^{4}H(y)\left(-\frac{1}{2x(y)}\right)dy}_{-\infty<x<0} + \underbrace{\int_{4}^{-\infty}H(y)\left(-\frac{1}{2x(y)}\right)dy}_{0<x<\infty}$$
$$= \int_0^4\left(-\frac{1}{-2\sqrt{4-y}}\right)dy - \int_0^4\left(-\frac{1}{2\sqrt{4-y}}\right)dy = \int_0^4\frac{1}{\sqrt{4-y}}dy = -2\sqrt{4-y}\Big|_{y=0}^4 = 4$$

oder $$\underbrace{\int_{-\infty}^{\infty}H(4-x^2)dx}_{H(4-x^2)=1 \text{ wenn } -2<x<2} = \int_{-2}^{2}dx = 4$$

e)
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H\left(R^2-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy = (\text{Fläche der Ellipse: } \frac{x^2}{a^2R^2}-\frac{y^2}{b^2R^2}<1) = \pi R^2 ab$$
oder
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H\left(R^2-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy = \int_{-aR}^{aR}2b\sqrt{R^2-\frac{x^2}{a^2}}dx = \frac{b}{a}\underbrace{\int_{-aR}^{aR}2\sqrt{a^2R^2-x^2}dx}_{\text{Fläche des Kreises mit Radius } aR} = \pi R^2 ab$$

f)
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta\left(E-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d}{dE}H\left(E-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy = \frac{d}{dE}\pi Eab = \pi ab$$
oder
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta\left(E-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy = \int_{-a\sqrt{E}}^{a\sqrt{E}}\left(\left.\frac{b^2}{|2y|}\right|_{y=-b\sqrt{E-\frac{x^2}{a^2}}} + \left.\frac{b^2}{|2y|}\right|_{y=b\sqrt{E-\frac{x^2}{a^2}}}\right)dx$$
$$= \int_{-a\sqrt{E}}^{a\sqrt{E}}\frac{b}{\sqrt{E-\frac{x^2}{a^2}}}dx = \int_{-a\sqrt{E}}^{a\sqrt{E}}\frac{ab}{\sqrt{a^2E-x^2}}dx = \int_{-\pi/2}^{\pi/2}\frac{ab}{\sqrt{a^2E-a^2E\sin^2 t}}a\sqrt{E}\cos t dt = ab\int_{-\pi/2}^{\pi/2}dt = \pi ab$$

7.2 Cauchyscher Hauptwert



a) $\int_{C_1} \frac{1-e^{iz}}{z^2} dz = \int_0^\pi \frac{1-e^{iRe^{i\theta}}}{R^2 e^{2i\theta}} iRe^{i\theta} d\theta = i \underbrace{\int_0^\pi \frac{1-e^{iRe^{i\theta}}}{Re^{i\theta}} d\theta}_{\text{wenn } \mathrm{Re}(ie^{i\theta})<0, e^{iRe^{i\theta}} \to 0} \xrightarrow{R\to\infty} 0$

b) $\int_{C_2} \frac{1-e^{iz}}{z^2} dz = \int_\pi^0 \frac{1-e^{ire^{i\theta}}}{r^2 e^{2i\theta}} ire^{i\theta} d\theta = i \int_\pi^0 \frac{1-e^{ire^{i\theta}}}{re^{i\theta}} d\theta$
$= -i \int_\pi^0 \frac{1}{re^{i\theta}} \sum_{n=1}^\infty \frac{i^n e^{in\theta}}{n!} r^n d\theta \xrightarrow{r\to 0} -i \int_\pi^0 \frac{1}{re^{i\theta}} \frac{ie^{i\theta}}{1!} r\, d\theta = \int_\pi^0 d\theta = -\pi$

c) $\mathcal{P} \int_{-\infty}^\infty \frac{1-e^{ix}}{x^2} dx = \lim_{\varepsilon\to 0^+} \left[\int_\varepsilon^\infty \frac{1-e^{ix}}{x^2} dx + \int_{-\infty}^{-\varepsilon} \frac{1-e^{ix}}{x^2} dx \right]$
$= \oint_C \frac{1-e^{iz}}{z^2} dz - \lim_{R\to\infty} \int_{C_1} \frac{1-e^{iz}}{z^2} dz - \lim_{r\to 0} \int_{C_2} \frac{1-e^{iz}}{z^2} dz = 0 - 0 + \pi = \pi$

d)
$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \to$ keine singularität
$\to \int_{-\infty}^\infty \frac{1-\cos x}{x^2} dx = \mathcal{P} \int_{-\infty}^\infty \frac{1-\cos x}{x^2} dx = \lim_{\varepsilon\to 0^+} \left[\int_\varepsilon^\infty \frac{1-\cos x}{x^2} dx + \int_{-\infty}^{-\varepsilon} \frac{1-\cos x}{x^2} dx \right]$
$= \lim_{\varepsilon\to 0^+} \left[\mathrm{Re} \int_\varepsilon^\infty \frac{1-e^{ix}}{x^2} dx + \mathrm{Re} \int_{-\infty}^{-\varepsilon} \frac{1-e^{ix}}{x^2} dx \right] = \mathrm{Re} \left[\mathcal{P} \int_{-\infty}^\infty \frac{1-e^{ix}}{x^2} dx \right] = \pi$
oder
$\mathcal{P} \int_{-\infty}^\infty \frac{1-\cos x}{x^2} dx = \frac{1}{2} \mathcal{P} \int_{-\infty}^\infty \frac{1-e^{ix}}{x^2} dx + \frac{1}{2} \mathcal{P} \int_{-\infty}^\infty \frac{1-e^{-ix}}{x^2} dx = \frac{1}{2}\pi + \frac{1}{2}\pi = \pi$
($\mathcal{P} \int_{-\infty}^\infty \frac{1-e^{-ix}}{x^2} dx$ kann mit dem Integrations Pfad vom unteren Halbkreis gerechnet werden.)

7.3 Delta-Distribution und Heaviside-Funktion

a) $\int_{-\infty}^\infty \frac{n}{2\cosh^2(nx)} \varphi(x) dx = \int_{-\infty}^\infty \frac{n}{2\cosh^2 y} \varphi\left(\frac{y}{n}\right) \frac{1}{n} dy = \int_{-\infty}^\infty \frac{1}{2\cosh^2 y} \varphi\left(\frac{y}{n}\right) dy$
$\xrightarrow{n\to\infty} \varphi(0) \int_{-\infty}^\infty \frac{1}{2\cosh^2 y} dy = \varphi(0) \frac{1}{2} \tanh y \Big|_{-\infty}^\infty = \varphi(0)$
b) $g_n(x) = \int_{-\infty}^x \frac{n}{2\cosh^2(nx')} dx' = \int_{-\infty}^{nx} \frac{n}{2\cosh^2 y} \frac{1}{n} dy = \int_{-\infty}^{nx} \frac{1}{2\cosh^2 y} dy = \frac{1}{2} \tanh y \Big|_{-\infty}^{nx} = \frac{1}{2} \tanh(nx) + \frac{1}{2}$
Im Limes $n \to 0$, wenn $x > 0$, $\tanh(nx) \to 1$ und $g_n(x) \to 1$. Wenn $x < 0$, $\tanh(nx) \to -1$ und $g_n(x) \to 0$.

Anmerkung: Integral von $1/\cosh^2(y)$
$\int_{-\infty}^{nx} \frac{1}{2\cosh^2 y} dy = \int_{-\infty}^{nx} \frac{2}{(e^y+e^{-y})^2} dy = \int_{-\infty}^{nx} \frac{2e^{2y}}{(e^{2y}+1)^2} dy = \int_0^{e^{2nx}} \frac{2u}{(u+1)^2} \frac{1}{2u} du = \int_0^{e^{2nx}} \frac{1}{(u+1)^2} du$
$= -\frac{1}{u+1}\Big|_{u=0}^{e^{2nx}} = -\frac{1}{e^{2nx}+1} + 1 = \frac{e^{nx}}{e^{nx}+e^{-nx}} \xrightarrow{n\to\infty} 1 \ (x > 0)$ oder $0 \ (x < 0)$

7.4 Entwicklung der Delta-Distribution

a)
Orthonormale Polynome: $\int_a^b \phi_n(x)\phi_m(x) dx = \delta_{nm}$
Entwicklung der Delta-Distribution: $\delta(x-t) = \sum_{n=0}^\infty a_n(t)\phi_n(x)$
$\int_a^b \delta(x-t)\phi_n(x) dx = \phi_n(t)$
oder wenn $\delta(x-t)$ durch $\sum_{m=0}^\infty a_m(t)\phi_m(x)$ ersetzt wird
$\int_a^b \delta(x-t)\phi_n(x) dx = \int_a^b \sum_{m=0}^\infty a_m(t)\phi_m(x)\phi_n(x) dx = \sum_{m=0}^\infty a_m(t)\delta_{nm} = a_n(t)$
Vergleich zwischen beiden Integrale: $a_n(t) = \phi_n(t)$
b)
$F(x) = \sum_{n=0}^N f_n\phi_n(x)$ und $\psi_i(x) = \sum_{n=0}^N \phi_n(x_i)\phi_n(x)$
$\int_a^b F(x)\psi_i(x) dx = \int_a^b \sum_{n=0}^N f_n\phi_n(x) \sum_{m=0}^N \phi_m(x_i)\phi_m(x) dx = \sum_{n=0}^N \sum_{m=0}^N f_n\phi_m(x_i) \int_a^b \phi_n(x)\phi_m(x) dx = \sum_{n=0}^N \sum_{m=0}^N f_n\phi_m(x_i)\delta_{nm} = \sum_{n=0}^N f_n\phi_n(x_i) = F(x_i)$

Anmerkung: Wenn $\{\psi_i(x) | i = 0, 1, 2, \cdots, N\}$ eine orthogonale Basis ist (d.h $\int_a^b \psi_i(x)\psi_j(x) dx = \sum_{n=0}^N \phi_n(x_i)\phi_n(x_j) = \lambda_i\delta_{ij}$), $F(x) = \sum_{i=0}^N \lambda_i^{-1} F(x_i)\psi_i(x) = \sum_{i=0}^N F(x_i)\psi_i^*(x)$ ($\psi_i^*(x)$: duale Basis, d.h. $\int_a^b \psi_i^*(x)\psi_j(x) dx = \delta_{ij}$).
Mit dieser Basis wird das Integral von 2 Funktionen $\int_a^b F(x)G(x) dx = \sum_{i=0}^N F(x_i)G(x_i)\lambda_i^{-1}$ als einer diskreten Summe geschrieben (Discrete Variable Representation).