1. Test - Lösungen **2.12.2016**

1 Rechenbeispiele [30 Punkte, 5 Punkte je Frage]

1) $\partial_i(x^i\sqrt{x^j x_j}) = \delta^i{}_i\sqrt{x^j x_j} + x^i\frac{1}{2}\frac{1}{\sqrt{x^k x_k}}(\delta^j{}_i x_j + \delta_{ji}x^j) = 3\sqrt{x^j x_j} + \frac{x^i x_i}{\sqrt{x^k x_k}} = 3\sqrt{x^j x_j} + \sqrt{x^i x_i} = 4\sqrt{x^i x_i}$

2) $(\partial_i x^i)(\partial_j x^j) + (\partial_i x^j)(\partial_j x^i) = \delta^i{}_i\delta^j{}_j + \delta^j{}_i\delta^i{}_j = \delta^i{}_i\delta^j{}_j + \delta^i{}_i = 12$

3) $\mathbf{e}^1 = C_1(1\ -2),\quad \mathbf{e}^1\cdot\mathbf{e}_1 = -3C_1 = 1 \to C_1 = -1/3 \to \mathbf{e}^1 = \frac{1}{3}(-1\ 2)$

$\mathbf{e}^2 = C_2(2\ -1),\quad \mathbf{e}^2\cdot\mathbf{e}_2 = 3C_2 = 1 \to C_2 = 1/3 \to \mathbf{e}^2 = \frac{1}{3}(2\ -1)$

alternative Lösung: $\begin{pmatrix}\mathbf{e}^1\\ \mathbf{e}^2\end{pmatrix} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{pmatrix}^{-1} = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}^{-1} = -\frac{1}{3}\begin{pmatrix}1 & -2\\ -2 & 1\end{pmatrix}$

4) $[\sigma_x,\sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} - \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$

$= 2i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

5) Da $\mathbf{g} = (g_{ij})$ und $\mathbf{g}^{-1} = (g^{ij})$, gilt $x^i{}_j = g^{ik}g_{kj} \to \mathbf{X} = \mathbf{g}^{-1}\mathbf{g} = \mathbf{I} \to \det(\mathbf{X}) = 1$

6) $\oint_C \frac{z}{2z+1}dz = 2\pi i\ \frac{z}{2}\Big|_{z=-1/2} = -\frac{1}{2}\pi i$

2 Spektraltheorem [30 Punkte]

a)

Eigenwertgleichung: $\mathbf{A}\begin{pmatrix}a\\ b\end{pmatrix} = \lambda_i\begin{pmatrix}a\\ b\end{pmatrix} \to \begin{pmatrix}-a+2b\\ 2a-b\end{pmatrix} = \lambda_i\begin{pmatrix}a\\ b\end{pmatrix} \to 2b^2 - 2a^2 = 0 \to b = \pm a$

Wenn $b = -a$, $\mathbf{A}\begin{pmatrix}a\\ -a\end{pmatrix} = \begin{pmatrix}-3a\\ 3a\end{pmatrix} = -3\begin{pmatrix}a\\ -a\end{pmatrix} \to \lambda_1 = -3$ und $\mathbf{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$ oder $-\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$

Wenn $b = a$, $\mathbf{A}\begin{pmatrix}a\\ a\end{pmatrix} = \begin{pmatrix}a\\ a\end{pmatrix} \to \lambda_2 = 1$ und $\mathbf{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$ oder $-\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$

b) $\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(t=0)$

Rechnung der Matrix $\exp(\mathbf{A}t)$:

$\mathbf{e}_1\cdot\mathbf{e}_2 = 0 \to \{\mathbf{e}_1,\mathbf{e}_2\}$ ist eine orthonormale Basis.

Spektraltheorem : $\mathbf{A} = -3\mathbf{E}_1 + \mathbf{E}_2$ mit $\mathbf{E}_1 = \mathbf{e}_1\otimes\mathbf{e}_1^T = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$ und $\mathbf{E}_2 = \mathbf{e}_2\otimes\mathbf{e}_2^T = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$

$\to \exp(\mathbf{A}t) = e^{-3t}\mathbf{E}_1 + e^t\mathbf{E}_2 \to \mathbf{x}(t) = e^{-3t}\mathbf{E}_1\begin{pmatrix}1\\ 0\end{pmatrix} + e^t\mathbf{E}_2\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{2}e^{-3t}\begin{pmatrix}1\\ -1\end{pmatrix} + \frac{1}{2}e^t\begin{pmatrix}1\\ 1\end{pmatrix}$

(Projektoren sind unabhägig vom Vorzeichen der Eigenvektoren, d.h. $\mathbf{E}_i = \mathbf{e}_i\otimes\mathbf{e}_i^T = (-\mathbf{e}_i)\otimes(-\mathbf{e}_i^T)$)

Alternative Lösung 1:

$\mathbf{U} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$

$\to \mathbf{U}\mathbf{U}^T = \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \to \{\mathbf{e}_1,\mathbf{e}_2\}$ ist eine orthonormale Basis.

$\mathbf{A} = \mathbf{U}\begin{pmatrix}-3 & 0\\ 0 & 1\end{pmatrix}\mathbf{U}^T \to \exp(\mathbf{A}t) = \mathbf{U}\begin{pmatrix}e^{-3t} & 0\\ 0 & e^t\end{pmatrix}\mathbf{U}^T$

$\mathbf{x}(t) = \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}e^{-3t} & 0\\ 0 & e^t\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}e^{-3t}\\ e^t\end{pmatrix} = \frac{1}{2}\begin{pmatrix}e^{-3t}+e^t\\ -e^{-3t}+e^t\end{pmatrix}$

Anmerkung: $\mathbf{U}$ ist abhängig vom Vorzeichen der Basisvektoren. Die mögliche Lösungen für $\mathbf{U}$ sind

$\mathbf{U} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}$ oder $\begin{pmatrix}-1 & -1\\ 1 & -1\end{pmatrix}$

Aber das Endergebnis $\mathbf{x}(t)$ ist unabhängig vom Vorzeichen.

Alternative Lösung 2:

$\mathbf{y} = \mathbf{U}^T\mathbf{x} = \frac{1}{\sqrt{2}}\begin{pmatrix}x_1 - x_2\\ x_1 + x_2\end{pmatrix} \to \frac{d}{dt}\mathbf{y} = \mathbf{U}^T\frac{d}{dt}\mathbf{x} = \underbrace{\mathbf{U}^T\mathbf{U}}_{=\mathbf{I}}\begin{pmatrix}-3 & 0\\ 0 & 1\end{pmatrix}\mathbf{U}^T\mathbf{x} = \begin{pmatrix}-3 & 0\\ 0 & 1\end{pmatrix}\mathbf{y}$

$$\mathbf{y}(t) = \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{t} \end{pmatrix} \mathbf{y}(t=0)$$

$$\mathbf{x}(t) = \mathbf{U}\mathbf{y}(t) = \mathbf{U} \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{t} \end{pmatrix} \mathbf{y}(t=0) = \mathbf{U} \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{t} \end{pmatrix} \mathbf{U}^T \mathbf{x}(t=0)$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-3t} \\ e^{t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{-3t} + e^{t} \\ -e^{-3t} + e^{t} \end{pmatrix}$$

3 Lokale Transformation [40 Punkte]

a)

Kartesische Basis: $\mathbf{e}_1 = \hat{\mathbf{x}}$, $\mathbf{e}_2 = \hat{\mathbf{y}}$, $\mathbf{e}_3 = \hat{\mathbf{z}}$.

Zylinderkoordinaten: $(x'^1, x'^2, x'^3) = (\rho, \theta, z)$

Transformation der Koordinaten: $x^1 = \rho \cos\theta$, $x^2 = \rho \sin\theta$, $x^3 = z$.

Transformation der Basis: $\mathbf{dx} = dx^i \mathbf{e}_i = dx'^j (\partial'_j x^i) \mathbf{e}_i = dx'^j \mathbf{e}'_j \to \mathbf{e}'_j = \partial'_j x^i \mathbf{e}_i \quad (\partial'_j x^i = \frac{\partial x^i}{\partial x'^j})$

$$\left. \begin{array}{l} \mathbf{e}'_1 = \partial'_1 x^i \mathbf{e}_i = \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2 \\ \mathbf{e}'_2 = \partial'_2 x^i \mathbf{e}_i = -\rho \sin\theta\, \mathbf{e}_1 + \rho\cos\theta\, \mathbf{e}_2 \\ \mathbf{e}'_3 = \partial'_3 x^i \mathbf{e}_i = \mathbf{e}_3 \end{array} \right\} \to \mathbf{e}'_i = \mathbf{e}_j t^j{}_i \text{ mit } t^i{}_j = \partial'_j x^i$$

$$\mathbf{T} = (t^i{}_j) = \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b)

$g'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j = t^k{}_i \mathbf{e}_k t^l{}_j \mathbf{e}_l = t^k{}_i t^l{}_j \delta_{kl} = t^k{}_i t^k{}_j \to \mathbf{g}' = \mathbf{T}^T \mathbf{T}$

$$\to \mathbf{g}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ und } \mathbf{g}'^* = \mathbf{g}'^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Alternative Lösung 1 für $\mathbf{g}'^*$:

$V = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = \rho$ Duale Basis: $\mathbf{e}_1 = \frac{1}{\rho}(\mathbf{e}_2 \times \mathbf{e}_3)^T = \mathbf{e}_1^T$, $\mathbf{e}_2 = \frac{1}{\rho}(\mathbf{e}_3 \times \mathbf{e}_1)^T = \frac{1}{\rho^2}\mathbf{e}_2^T$, $\mathbf{e}_3 = \frac{1}{\rho}(\mathbf{e}_1 \times \mathbf{e}_2)^T = \mathbf{e}_3^T$

$$g'^{ij} = \mathbf{e}'^i \cdot \mathbf{e}'^j \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Alternative Lösung 2 für $\mathbf{g}'^*$:

Duale Basis: $\mathbf{g}'$ diagonal $\to \mathbf{e}'_i$ orthogonale Basis $\to \mathbf{e}^i = C_i \mathbf{e}_i^T$.

Normierung $\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i{}_j \to \mathbf{e}^1 = \mathbf{e}_1^T$, $\mathbf{e}^2 = \frac{1}{\rho^2}\mathbf{e}_2^T$, $\mathbf{e}^3 = \mathbf{e}_3^T$. $\to g^{ij} = \mathbf{e}'^i \cdot \mathbf{e}'^j \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c)

$dV = \mathbf{e}'_1 \cdot (\mathbf{e}'_2 \times \mathbf{e}'_3) d\rho d\theta dz = \mathbf{e}'_1 \cdot (\rho\cos\theta \mathbf{e}_2 + \rho \sin\theta \mathbf{e}'_3) d\rho d\theta dz = \rho d\rho d\theta dz$

(oder $dV = \sqrt{\det(\mathbf{g}')} d\rho d\theta dz = \rho d\rho d\theta dz$)

$F_1 : dr = 0 \to d\mathbf{F}_1 = \mathbf{e}'_2 \times \mathbf{e}'_3 d\theta dz = \mathbf{e}'_1 \rho|_{\rho=R} d\theta dz = R\mathbf{e}'_1 d\theta dz$

$F_2 : dz = 0 \to d\mathbf{F}_2 = \mathbf{e}'_1 \times \mathbf{e}'_2 d\theta d\rho = \mathbf{e}'_3 \rho d\theta d\rho$

$F_3 : dz = 0 \to d\mathbf{F}_3 = \mathbf{e}'_2 \times \mathbf{e}'_1 d\theta d\rho = -\mathbf{e}'_3 \rho d\theta d\rho$ (ein äußeres Normalenvektor)

d)

Gaußscher Integralsatz :

$\int_F \mathbf{w} \cdot d\mathbf{F} = \int_V \nabla \cdot \mathbf{w} dV$ mit $\nabla \cdot \mathbf{w} = y - 1 + 1 = \rho \sin\theta$

$\to \int_V \nabla \cdot \mathbf{w} dV = \int_0^R d\rho \int_0^{2\pi} d\theta \int_0^h dz \rho^2 \sin\theta = \int_0^R \rho^2 d\rho \int_0^{2\pi} \sin\theta d\theta \int_0^h dz = 0$

Alternative Lösung:

$\mathbf{w} = \rho^2 \cos\theta \sin\theta \mathbf{e}_1 - \rho \sin\theta \mathbf{e}_2 + z\mathbf{e}_3$

$\int_{F_1} \mathbf{w} \cdot d\mathbf{F} = \int_0^h \int_0^{2\pi} \mathbf{w}|_{\rho=R} \cdot \mathbf{e}'_1 R d\theta dz = \int_0^h \int_0^{2\pi} (R^2 \cos^2\theta \sin\theta - R\sin^2\theta) R d\theta dz = -hR^2\pi$

$\int_{F_2} \mathbf{w} \cdot d\mathbf{F} = \int_0^R \int_0^{2\pi} \mathbf{w}|_{z=h} \cdot \mathbf{e}'_3 \rho d\theta d\rho = \int_0^R \int_0^{2\pi} z|_{z=h} \rho d\theta d\rho = \pi h R^2$

$\int_{F_3} \mathbf{w} \cdot d\mathbf{F} = \int_0^R \int_0^{2\pi} \mathbf{w}|_{z=0} \cdot (-\mathbf{e}'_3) \rho d\theta d\rho = -\int_0^R \int_0^{2\pi} z|_{z=0} \rho d\theta d\rho = 0$

$\int_F \mathbf{w} \cdot d\mathbf{F} = \int_{F_1} \mathbf{w} \cdot d\mathbf{F} + \int_{F_2} \mathbf{w} \cdot d\mathbf{F} + \int_{F_3} \mathbf{w} \cdot d\mathbf{F} = 0$