10. Tutorium - Lösungen **13.1.2017**

- ANMERKUNG: Es liegt in der Verantwortung des Einzelnen, sich die Beispiele zunächst alleine und ganz ohne Hilfsmittel anzuschauen. Google, Wolfram Alpha, Lösungssammlungen, etc. helfen nur kurzfristig - leider nicht beim Test!

10.1 Gamma-Funktion

a) $|(ix)!|^2 = |\Gamma(1+ix)|^2 = \Gamma(1+ix)\Gamma(1-ix) = \Gamma(1+ix)(-ix)\Gamma(-ix) = -ix\frac{\pi}{\sin(-i\pi x)} = 2x\frac{\pi}{e^{\pi x}-e^{-\pi x}} = \frac{\pi x}{\sinh(\pi x)}$

b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^2+y^2+z^2)^{n/2}(\pi)^{-3/2}e^{-x^2-y^2-z^2}dxdydz = \int_0^\infty dr\int_0^\pi d\theta\int_0^{2\pi}d\phi r^2\sin\theta r^n(\pi)^{-3/2}e^{-r^2}$
$= \frac{4}{\sqrt{\pi}}\int_0^\infty drr^2r^ne^{-r^2} = \frac{4}{\sqrt{\pi}}\int_0^\infty dt\frac{1}{2t^{1/2}}t^{n/2+1}e^{-t} = \frac{2}{\sqrt{\pi}}\int_0^\infty dtt^{(n+1)/2}e^{-t} = \frac{2}{\sqrt{\pi}}\Gamma((n+3)/2)$

c) $\int_a^\infty \exp(2ax-x^2)dx = \exp(a^2)\int_a^\infty\exp(-(x-a)^2)dx = \exp(a^2)\int_0^\infty\exp(-t)\frac{1}{2\sqrt{t}}dt = \frac{1}{2}\Gamma(1/2)\exp(a^2) = \frac{1}{2}\sqrt{\pi}\exp(a^2)$

d) $I_n \equiv \int_0^{\pi/2}\sin^n\theta d\theta$
Wenn $n=0$, $I_0=\pi/2$.
Wenn $n=1$, $I_1=1$.
Wenn $n>1$
$I_n = -\cos\theta\sin^{n-1}\theta\Big|_0^{\pi/2} + (n-1)\int_0^{\pi/2}\cos^2\theta\sin^{n-2}\theta d\theta$
$= (n-1)\int_0^{\pi/2}\sin^{n-2}\theta d\theta - (n-1)\int_0^{\pi/2}\sin^n\theta d\theta = (n-1)I_{n-2}-(n-1)I_n$
$\to I_n = \frac{n-1}{n}I_{n-2} = \frac{n-1}{n}\frac{n-3}{n-2}I_{n-4} = \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}I_{n-6} = \frac{(n-1)/2}{n/2}\frac{(n-1)/2-1}{n/2-1}\frac{(n-1)/2-2}{n/2-2}I_{n-6} = \cdots$
Für gerade n
$I_n = \frac{(n-1)/2}{n/2}\frac{(n-1)/2-1}{n/2-1}\frac{(n-1)/2-2}{n/2-2}\cdots\frac{1/2}{2/2}I_0 = \frac{\Gamma((n+1)/2)/\Gamma(1/2)}{\Gamma(n/2+1)/\Gamma(1)}\frac{\pi}{2} = \frac{\Gamma((n+1)/2)}{\Gamma(n/2+1)}\frac{\sqrt{\pi}}{2}$
Für ungerade n
$I_n = \frac{(n-1)/2}{n/2}\frac{(n-1)/2-1}{n/2-1}\frac{(n-1)/2-2}{n/2-2}\cdots\frac{2/2}{3/2}I_1 = \frac{\Gamma((n+1)/2)/\Gamma(1)}{\Gamma(n/2+1)/\Gamma(3/2)} = \frac{\Gamma((n+1)/2)}{\Gamma(n/2+1)}\frac{1}{2/\Gamma(1/2)} = \frac{\Gamma((n+1)/2)}{\Gamma(n/2+1)}\frac{\sqrt{\pi}}{2}$

e) $I_n = \frac{\Gamma((n+1)/2)}{\Gamma(n/2+1)}\frac{\sqrt{\pi}}{2} = \frac{\Gamma((n+1)/2)\Gamma(1/2)}{\Gamma(n/2+1)}\frac{1}{2} = \frac{1}{2}B((n+1)/2,1/2)$

Alternative Lösung für d) und e):
$t=\cos^2\theta \to I_n = \int_0^{\pi/2}\sin^n\theta d\theta = \int_0^1(1-t)^{n/2}\frac{1}{2}t^{-1/2}(1-t)^{-1/2}dt = \frac{1}{2}\int_0^1(1-t)^{(n-1)/2}t^{-1/2}dt$
$= \frac{1}{2}B((n+1)/2,1/2) \to I_n = \frac{1}{2}\frac{\Gamma((n+1)/2)\Gamma(1/2)}{\Gamma(n/2+1)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma((n+1)/2)}{\Gamma(n/2+1)}$
Anmerkung : Rekursionsrelation für das Volmen $V_n(R)$ einer n-dimensionalen Kugel mit Radius R
$V_n(R) = RB((n+1)/2,1/2)V_{n-1}(R) \to$ Lösung $V_n(R) = R^n\frac{\pi^{n/2}}{\Gamma(n/2+1)}$

10.2 Diffusionsgleichung

a) $(\partial_t - \frac{1}{2}D\partial_x^2)G(t,t';x,x') = \delta(t-t')\delta(x-x')$
$G(t,t';x,x') = \frac{1}{(2\pi)^2}\int_{-\infty}^\infty d\omega\int_{-\infty}^\infty dk\tilde{G}(\omega,k)e^{i\omega(t-t')}e^{ik(x-x')}$
$\to (\partial_t - \frac{1}{2}D\partial_x^2)G(t,t';x,x') = \frac{1}{4\pi^2}\int_{-\infty}^\infty d\omega\int_{-\infty}^\infty dk(i\omega+\frac{1}{2}Dk^2)\tilde{G}(\omega,k)e^{i\omega(t-t')}e^{ik(x-x')}$
und $\delta(t-t')\delta(x-x') = \frac{1}{4\pi^2}\int_{-\infty}^\infty d\omega\int_{-\infty}^\infty dke^{i\omega(t-t')}e^{ik(x-x')}$
Vergleich der Integranden: $(i\omega+\frac{1}{2}Dk^2)\tilde{G}(\omega,k) = 1 \to \tilde{G}(\omega,k) = \frac{1}{i\omega+\frac{1}{2}Dk^2}$
$G(t,t';x,x') = \frac{1}{2\pi}\int_{-\infty}^\infty dk\, e^{ik(x-x')}\frac{1}{2\pi}\int_{-\infty}^\infty d\omega\frac{1}{i\omega+\frac{1}{2}Dk^2}e^{i\omega(t-t')}$
$= \frac{1}{2\pi}\int_{-\infty}^\infty dk\, e^{ik(x-x')}\frac{1}{2\pi i}\int_{-\infty}^\infty d\omega\frac{1}{\omega-i\frac{1}{2}Dk^2}e^{i\omega(t-t')}$
$= \frac{1}{2\pi}\int_{-\infty}^\infty dk\, e^{ik(x-x')}H(t-t')e^{-\frac{1}{2}Dk^2(t-t')}$
$= \frac{1}{2\pi}H(t-t')\int_{-\infty}^\infty dk\,\exp\left(-\frac{1}{2}D(t-t')\left(k-i\frac{x-x'}{D(t-t')}\right)^2-\frac{(x-x')^2}{2D(t-t')}\right)$
$= H(t-t')\sqrt{\frac{1}{2\pi D(t-t')}}\exp\left(-\frac{(x-x')^2}{2D(t-t')}\right)$

Alternative Lösung :
$G(t,t';x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\tilde{G}(t,t';k)e^{ik(x-x')}$
$\to \partial_t\tilde{G}(t,t';k) + \frac{1}{2}Dk^2\tilde{G}(t,t';k) = \delta(t-t')$
Ansatz : $\tilde{G}(t,t';k) = F(t-t')e^{-(1/2)Dk^2(t-t')}$
$\to F'(t-t')e^{-(1/2)Dk^2(t-t')} = \delta(t-t') \to F'(t-t') = \delta(t-t')e^{(1/2)Dk^2(t-t')} = \delta(t-t') \to F(t-t') = H(t-t')$
$G(t,t';x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dkH(t-t')e^{-(1/2)Dk^2(t-t')}e^{ik(x-x')} = H(t-t')\sqrt{\frac{1}{2\pi D(t-t')}}\exp\left(-\frac{(x-x')^2}{2D(t-t')}\right)$
b)
$y(t,x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(t,t';x,x')f(t',x')dt'dx'$
$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(t-t')\sqrt{\frac{1}{2\pi D(t-t')}}\exp\left(-\frac{(x-x')^2}{2D(t-t')}\right)\delta(t')\frac{1}{\sqrt{2\pi}\sigma}e^{-x'^2/(2\sigma^2)}dt'dx'$
$= H(t)\sqrt{\frac{1}{2\pi Dt}}\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-x')^2}{2Dt}\right)e^{-x'^2/(2\sigma^2)}dx'$
$= H(t)\sqrt{\frac{1}{2\pi Dt}}\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{Dt+\sigma^2}{2Dt\sigma^2}\left(x' - \frac{\sigma^2}{\sigma^2+Dt}x\right)^2 - \frac{1}{2(\sigma^2+Dt)}x^2\right)dx'$
$= H(t)\sqrt{\frac{1}{2\pi(Dt+\sigma^2)}}\exp\left(-\frac{1}{2(\sigma^2+Dt)}x^2\right)$

10.3 Frobenius-Methode

a) Ansatz : $y(x) = \sum_{n=0}^{\infty} a_n x^{n+\sigma}$ mit $a_0 \neq 0$
$4xy'' + 2y' + y = \sum_{n=0}^{\infty} x^\sigma\left[4(n+\sigma)(n+\sigma-1)a_n x^{n-1} + 2(n+\sigma)a_n x^{n-1} + a_n x^n\right]$
$= x^\sigma\sum_{n=-1}^{\infty}\left[4(n+\sigma+1)(n+\sigma) + 2(n+\sigma+1)\right]a_{n+1}x^n + x^\sigma\sum_{n=0}^{\infty} a_n x^n$
Die Gleichung $x^\sigma\sum_{n=-1}^{\infty}\left[4(n+\sigma+1)(n+\sigma) + 2(n+\sigma+1)\right]a_{n+1}x^n + x^\sigma\sum_{n=0}^{\infty} a_n x^n = 0$ gilt für beliebige x.
$\to [4(n+\sigma+1)(n+\sigma) + 2(n+\sigma+1)]a_{n+1} + a_n = 0$ für $n \geq 0$
und $[4(n+\sigma+1)(n+\sigma) + 2(n+\sigma+1)]\,a_{n+1} = 0$ für $n = -1$.
Für $n = -1$, gilt $[4\sigma(\sigma-1) + 2\sigma]\,a_0 = 0$ Da $a_0 \neq 0$, gilt $\sigma = 0, 1/2$
b) Wenn $\sigma = 0$ und $n \geq 0$,
$[4n(n+1) + 2(n+1)]a_{n+1} + a_n = 0 \to a_{n+1} = -\frac{1}{(2n+1)(2n+2)}a_n$
$\to a_1 = -\frac{1}{2}a_0,\quad a_2 = -\frac{1}{3\cdot 4}a_1 = \frac{1}{4!}a_0,\quad a_3 = -\frac{1}{5\cdot 6}a_2 = -\frac{1}{6!}a_0,\quad \cdots,\quad a_n = (-1)^n\frac{1}{(2n)!}a_0,$
Wenn $\sigma = 1/2$ und $n \geq 0$,
$[4(n+3/2)(n+1/2) + 2(n+3/2)]a_{n+1} + a_n = 0 \to a_{n+1} = -\frac{1}{(2n+2)(2n+3)}a_n$
$\to a_1 = -\frac{1}{2\cdot 3}a_0,\quad a_2 = -\frac{1}{4\cdot 5}a_1 = \frac{1}{5!}a_0,\quad a_3 = -\frac{1}{6\cdot 7}a_2 = -\frac{1}{7!}a_0,\quad \cdots,\quad a_n = (-1)^n\frac{1}{(2n+1)!}a_0,$
Allgemeine Lösung:
$y(x) = \tilde{c}_1\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}a_0(\sigma=0)x^n + \tilde{c}_2\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}a_0(\sigma=1/2)x^{n+1/2}$
$= c_1\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}x^n + c_2\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{n+1/2} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}x^n\left(c_1 + \frac{c_2}{2n+1}x^{1/2}\right)$
$(c_1 = \tilde{c}_1 a_0(\sigma=0),\quad c_2 = \tilde{c}_2 a_0(\sigma=1/2))$