7. Tutorium - Lösungen **25.11.2016**

- ANMERKUNG: Es liegt in der Verantwortung des Einzelnen, sich die Beispiele zunächst alleine und ganz ohne Hilfsmittel anzuschauen. Google, Wolfram Alpha, Lösungssammlungen, etc. helfen nur kurzfristig - leider nicht beim Test!

7.1 Residuensatz

a) $\oint_C \frac{z}{z^2-z-2}dz = \oint_C \frac{z}{(z-2)(z+1)}dz = 2\pi i \left(\frac{z}{z+1}\Big|_{z=2} + \frac{z}{z-2}\Big|_{z=-1} \right) = 2\pi i \left(\frac{2}{3}+\frac{1}{3}\right) = 2\pi i$

b) $\oint_C \frac{1}{2z^2-4z-16}dz = \oint_C \frac{1}{2(z-4)(z+2)}dz = 2\pi i \, \frac{1}{2(z-4)}\Big|_{z=-2} = 2\pi i \left(-\frac{1}{12}\right) = -\frac{1}{6}\pi i$

c) $\oint_C \frac{z^3+z^2+1}{2(z+1)^3}dz = \oint_C \frac{1}{2(z+1)^3}\left[1+(z+1)+\frac{1}{2}(-4(z+1)^2)+\frac{1}{3!}6(z+1)^3\right]dz$

$= \oint_C \left[\frac{1}{2(z+1)^3}+\frac{1}{2(z+1)^2}-\frac{1}{z+1}+\frac{1}{2}\right] = -2\pi i$

oder $\oint_C \frac{z^3+z^2+1}{2(z+1)^3}dz = 2\pi i \frac{1}{2!}\,\frac{d^2}{dz^2}\frac{1}{2}(z^3+z^2+1)\Big|_{z=-1} = 2\pi i \frac{1}{2!}\,(3z+1)|_{z=-1} = -2\pi i$

d) $\oint_C \frac{e^{2z}}{z(z-1)}dz = 2\pi i \left(\frac{e^{2z}}{z-1}\Big|_{z=0} + \frac{e^{2z}}{z}\Big|_{z=1} \right) = 2\pi i(-1+e^2)$

7.2 Gaußscher Integralsatz

a) Kugelfläche $F : r = R$ (konstant) $\to dx'^1 = dr = 0 \to$ Basisvektoren : $\mathbf{e}'_2 = \mathbf{e}'_\theta$ und $\mathbf{e}'_3 = \mathbf{e}'_\phi$
Vektor in der Tangentialebene
$\mathbf{r}_T - \mathbf{r} = (\theta_T-\theta)\partial_\theta \mathbf{r} + (\phi_T-\phi)\partial_\phi \mathbf{r} = (\theta_T-\theta)\partial_\theta x^i \mathbf{e}_i + (\phi_T-\phi)\partial_\phi x^i\mathbf{e}_i = (\theta_T-\theta)\mathbf{e}_\theta + (\phi_T-\phi)\mathbf{e}_\phi$
Aus Bsp. 6.3:
$\mathbf{e}_\theta = \mathbf{e}'_2 = \partial'_2 x^i \mathbf{e}_i = R\cos\theta\cos\phi\,\mathbf{e}_1 + R\cos\theta\sin\phi\,\mathbf{e}_2 - R\sin\theta\,\mathbf{e}_3$
$\mathbf{e}_\phi = \mathbf{e}'_3 = \partial'_3 x^i \mathbf{e}_i = -R\sin\theta\sin\phi\,\mathbf{e}_1 + R\sin\theta\cos\phi\,\mathbf{e}_2$

Alternative Lösung:
$r = R \to x^2+y^2+z^2 = R^2 \to z = \pm\sqrt{R^2-x^2-y^2} \equiv f_\pm(x,y) \to \mathbf{r}(x,y) = x\mathbf{e}_x + y\mathbf{e}_y + f_\pm(x,y)\mathbf{e}_3$
Tangentialebene $\mathbf{r}-\mathbf{r}_0 = (x_T-x)\partial_x\mathbf{r} + (y_T-y)\partial_y\mathbf{r}$
$\partial_x \mathbf{r}|_{r=R} = \mathbf{e}_1 + \partial_x f_\pm(x,y)\mathbf{e}_3 = \mathbf{e}_1 - (x/z)\mathbf{e}_3 = \mathbf{e}_1 - \tan\theta\cos\phi\mathbf{e}_3 = (R\cos\theta)^{-1}\cos\phi\,\mathbf{e}_\theta - (R\sin\theta)^{-1}\sin\phi\,\mathbf{e}_\phi$
$\partial_y \mathbf{r}|_{r=R} = \mathbf{e}_2 + \partial_y f_\pm(x,y)\mathbf{e}_3 = \mathbf{e}_2 - (y/z)\mathbf{e}_3 = \mathbf{e}_2 - \tan\theta\sin\phi\mathbf{e}_3 = (R\cos\theta)^{-1}\sin\phi\,\mathbf{e}_\theta + (R\sin\theta)^{-1}\cos\phi\,\mathbf{e}_\phi$
2 unabhängige Linearkombinationen von $\mathbf{e}_\theta$ und $\mathbf{e}_\phi$ können die Basisvektoren der Tangentialebene sein.

b) $dF = |(dx'^2\mathbf{e}'_2)\times(dx'^3\mathbf{e}'_3)| = dx'^2dx'^3|\mathbf{e}'_2\times\mathbf{e}'_3|$
$= dx'^2dx'^3|R^2\cos\theta\sin\theta\cos^2\phi\,\mathbf{e}_3 + R^2\cos\theta\sin\theta\sin^2\phi\,\mathbf{e}_3 + R^2\sin^2\theta\sin\phi\,\mathbf{e}_2 + R^2\sin^2\theta\cos\phi\,\mathbf{e}_1|$
$= dx'^2dx'^3|R^2\sin^2\theta\cos\phi\,\mathbf{e}_1 + R^2\sin^2\theta\sin\phi\,\mathbf{e}_2 + R^2\cos\theta\sin\theta\,\mathbf{e}_3| = dx'^2dx'^3R^2\sin\theta$

Aus Bsp.6.3 $(g'_{ij}) = (\mathbf{e}'_i\cdot\mathbf{e}'_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2\sin^2\theta \end{pmatrix} \to \det(\tilde{\mathbf{g}}') = g'_{22}g'_{33} - g'_{23}g'_{32} = R^4\sin^2\theta$

$dF = \sqrt{\det(\tilde{\mathbf{g}}')}d\theta d\phi$

Alternative Lösung:
Transformationsmatrix $\mathbf{T} = t^i{}_j$, Basistransformation $\mathbf{e}'_j = t^i{}_j\mathbf{e}_i$
$dF = dx'^2dx'^3|\mathbf{e}'_2\times\mathbf{e}'_3| = dx'^2dx'^3|(t^i{}_2\mathbf{e}_i)\times(t^j{}_3\mathbf{e}_j)| = dx'^2dx'^3|t^i{}_2t^j{}_3\varepsilon_{ij}{}^k\mathbf{e}_k|$
$= dx'^2dx'^3\sqrt{(\varepsilon_{ij}{}^kt^i{}_2t^j{}_3\mathbf{e}_k)\cdot(\varepsilon_{lm}{}^nt^l{}_2t^m{}_3\mathbf{e}_n)} = dx'^2dx'^3\sqrt{\varepsilon_{ij}{}^k\varepsilon_{lm}{}^nt^i{}_2t^j{}_3t^l{}_2t^m{}_3\delta_{kn}}$
$= dx'^2dx'^3\sqrt{\varepsilon_{ij}{}^k\varepsilon_{lmk}t^i{}_2t^j{}_3t^l{}_2t^m{}_3} = dx'^2dx'^3\sqrt{(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})t^i{}_2t^j{}_3t^l{}_2t^m{}_3}$
$= dx'^2dx'^3\sqrt{t^i{}_2t^j{}_3t_{i2}t_{j3} - t^i{}_2t^j{}_3t_{j2}t_{i3}} = dx'^2dx'^3\sqrt{t^i{}_2t_{i2}t^j{}_3t_{j3} - t^i{}_2t_{i3}t^j{}_3t_{j2}}$

$g'_{ij} = \mathbf{e}'_i\cdot\mathbf{e}'_j = (t^k{}_i\mathbf{e}_k)\cdot(t^l{}_j\mathbf{e}_l) = t^k{}_it^l{}_j\delta_{kl} = t^k{}_it_{kj}$ (für orthonormale Basis, $g_{kl} = \delta_{kl}$ und $t^k{}_j = t_{kj}$)
[Anmerkung : In der Transformationmatrix $t^k{}_j$ zwischen den 2 Basen ist k der Index der kartesischen Koordinaten und j der Index der Kugelkoordinaten (orthogonal aber nicht-normiert). Deswegen $t^k{}_j = t_{kj}$ aber $t_{kj} = t_k{}^lg'_{lj} \neq t_k{}^j$.]
$\det(\tilde{\mathbf{g}}') = g'_{22}g'_{33} - g'_{23}g'_{32} = t^k{}_2t_{k2}t^l{}_3t_{l3} - t^k{}_2t_{k3}t^l{}_3t_{l2} \to dF = \sqrt{\det(\tilde{\mathbf{g}}')}dx'_2dx'_3 = \sqrt{\det(\tilde{\mathbf{g}}')}d\theta d\phi.$

c) Normalenvektor: $\mathbf{n} = \frac{\mathbf{e}'_2 \times \mathbf{e}'_3}{|\mathbf{e}'_2 \times \mathbf{e}'_3|}$
$\rightarrow d\mathbf{F} = \mathbf{n}\, dF = \frac{\mathbf{e}'_2 \times \mathbf{e}'_3}{|\mathbf{e}'_2 \times \mathbf{e}'_3|} |\mathbf{e}'_2 \times \mathbf{e}'_3| d\theta d\phi = \mathbf{e}'_2 \times \mathbf{e}'_3 d\theta d\phi$
$= (R^2 \sin^2\theta \cos\phi\, \mathbf{e}_1 + R^2 \sin^2\theta \sin\phi\, \mathbf{e}_2 + R^2 \cos\theta \sin\theta\, \mathbf{e}_3) d\theta d\phi$
$= R^2 \sin\theta (\sin\theta\cos\phi\, \mathbf{e}_1 + \sin\theta \sin\phi\, \mathbf{e}_2 + \cos\theta\, \mathbf{e}_3) d\theta d\phi = R^2 \sin\theta \mathbf{e}'_1$
$\mathbf{w} = 2R \sin\theta\cos\phi \mathbf{e}_1 + R\cos\theta \mathbf{e}_3$
$\int_F \mathbf{w}\cdot d\mathbf{F} = \int_0^\pi d\theta \int_0^{2\pi} d\phi (2R^3 \sin^3\theta \cos^2\phi + R^3 \cos^2\theta \sin\theta) = \int_0^\pi d\theta (2\pi R^3 \sin^3\theta + 2\pi R^3 \cos^2\theta \sin\theta)$
$= 2\pi R^3 \int_0^\pi d\theta \sin\theta = 4\pi R^3$

d) Volumen des Parallelepipeds (siehe Bsp.3.3 und 4.1) : $dV = dx'^1 dx'^2 dx'^3 |\mathbf{e}'_1 \cdot (\mathbf{e}'_2 \times \mathbf{e}'_3)| = dx'^1 dx'^2 dx'^3 |\det(\mathbf{E}')|$
wobei $\mathbf{E}' = \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{pmatrix}$
Metrischer Tensor : $\mathbf{g}' = \mathbf{E}'^T \mathbf{E}' \rightarrow \det(\mathbf{g}') = \det(\mathbf{E}'^T \mathbf{E}') = \det(\mathbf{E}'^T)\det(\mathbf{E}') = \det(\mathbf{E}')^2$
$\rightarrow dV = dx'^1 dx'^2 dx'^3 \sqrt{\det(\mathbf{g}')} = r^2 \sin\theta dr d\theta d\phi$
e) $\nabla\cdot\mathbf{w} = \partial_x(2r\sin\theta\cos\phi) + \partial_z(r\cos\theta) = \partial_x(2x) + \partial_z(z) = 2 + 1 = 3$
$\int_V \nabla\cdot\mathbf{w} dV = \int_0^R \int_0^\pi \int_0^{2\pi} 3r^2 \sin\theta d\phi d\theta dr = 6\pi \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta = 6\pi \frac{R^3}{3} 2 = 4\pi R^3$

Gaußscher Integralsatz : $\int_V \nabla\cdot\mathbf{w} dV = \int_F \mathbf{w}\cdot d\mathbf{F}$

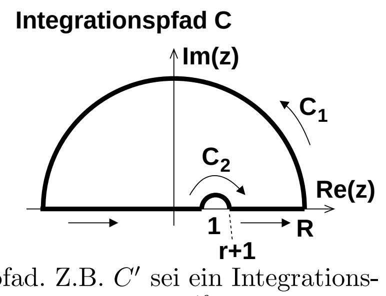
Alternative Lösung:
$\mathbf{w} = 2r\sin\theta\cos\phi\mathbf{e}_1 + r\cos\theta\mathbf{e}_3$
$\nabla\cdot\mathbf{w} = \mathbf{e}'^i \cdot \partial'_i \mathbf{w} = \mathbf{e}'^1 \cdot (2\sin\theta\cos\phi\mathbf{e}_1 + \cos\theta\mathbf{e}_3) + \mathbf{e}'^2 \cdot (2r\cos\theta\cos\phi\mathbf{e}_1 - r\sin\theta\mathbf{e}_3) + \mathbf{e}'^3(-2r\sin\theta\sin\phi\mathbf{e}_1)$
$\mathbf{e}'^i = g^{ij}\mathbf{e}'_j \rightarrow \mathbf{e}'^1 = \mathbf{e}'_1 = \sin\theta\cos\phi\,\mathbf{e}_1 + \sin\theta\sin\phi\,\mathbf{e}_2 + \cos\theta\,\mathbf{e}_3$
$\mathbf{e}'^2 = \frac{1}{r^2}\mathbf{e}'_2 = \frac{1}{r}\cos\theta\cos\phi\,\mathbf{e}_1 + \frac{1}{r}\cos\theta\sin\phi\,\mathbf{e}_2 - \frac{1}{r}\sin\theta\,\mathbf{e}_3, \quad \mathbf{e}'^3 = \frac{1}{r^2\sin^2\theta}\mathbf{e}'_3 = -\sin\phi/(r\sin\theta)\,\mathbf{e}_1 + \cos\phi/(r\sin\theta)\,\mathbf{e}_2$
$\rightarrow \nabla\cdot\mathbf{w} = 2\sin\theta^2\cos^2\phi + \cos^2\theta + 2\cos^2\theta\cos^2\phi + \sin^2\theta + 2\sin^2\phi = 3$
$\int_V \nabla\cdot\mathbf{w} dV = \int_V 3dV = 4\pi R^3$

7.3 Cauchyscher Hauptwert

a) $\int_{C_1} \frac{1}{z^3 - z^2 + z - 1} dz = \int_{C_1} \frac{1}{(z-1)(z+i)(z-i)} dz = \int_0^\pi \frac{1}{(Re^{i\theta}-1)(Re^{i\theta}+i)(Re^{i\theta}-i)} iRe^{i\theta} d\theta$
$\left|\int_0^\pi \frac{1}{(Re^{i\theta}-1)(Re^{i\theta}+i)(Re^{i\theta}-i)} iRe^{i\theta} d\theta\right| \leq \int_0^\pi \left|\frac{1}{(Re^{i\theta}-1)(Re^{i\theta}+i)(Re^{i\theta}-i)} iRe^{i\theta}\right| d\theta = \int_0^\pi \left|\frac{1}{Re^{i\theta}-1}\right| \left|\frac{1}{Re^{i\theta}+i}\right| \left|\frac{1}{Re^{i\theta}-i}\right| R d\theta$
$\leq \int_0^\pi \left|\frac{1}{R-1}\right| \left|\frac{1}{-iR+i}\right| \left|\frac{1}{iR-i}\right| R d\theta = \frac{R}{|R-1|^3} \int_0^\pi d\theta = 2\pi \frac{R}{|R-1|^3} \overset{R\to\infty}{\longrightarrow} 0$

$\lim_{R\to\infty} \int_{C_1} \frac{1}{z^3-z^2+z-1} dz = 0$
b) $\lim_{r\to 0} \int_{C_2} \frac{1}{z^3-z^2+z-1} dz = \lim_{r\to 0} \oint_{C_2} \frac{1}{(z-1)(z+i)(z-i)} dz = \underbrace{\lim_{r\to 0} \int_0^\pi \frac{1}{re^{i\theta}(re^{i\theta}+1+i)(re^{i\theta}+1-i)} i\, r\, e^{i\theta} d\theta}_{z = re^{i\theta}+1}$

$= \lim_{r\to 0} \int_0^\pi \frac{1}{(re^{i\theta}+1+i)(re^{i\theta}+1-i)} i d\theta = \int_0^\pi \frac{i}{(1+i)(1-i)} d\theta = i\frac{\pi}{2}$



c) $\mathcal{P}\int_{-\infty}^{\infty} \frac{1}{x^3-x^2+x-1} dx$
$= \oint_C \frac{1}{z^3-z^2+z-1} dz - \int_{C_1} \frac{1}{z^3-z^2+z-1} dz + \int_{C_2} \frac{1}{z^3-z^2+z-1} dz$
$= \oint_C \frac{1}{(z-1)(z+i)(z-i)} dz + i\frac{\pi}{2}$
$= 2\pi i \left.\frac{1}{(z-1)(z+i)}\right|_{z=i} + i\frac{\pi}{2} = \frac{\pi}{i-1} + i\frac{\pi}{2} = \frac{\pi(-i-1)}{2} + i\frac{\pi}{2} = -\frac{\pi}{2}$

Anmerkung : Der Cauchysche Hauptwert ist unabhängig von Integrationspfad. Z.B. C' sei ein Integrationspfad wie C aber der obere Halbkreis C_2 ist durch dem unteren Halbkreis $C'_2 = \{z = re^{i\theta} | \pi \leq \theta \leq 2\pi\}$ ersetzt.
$\lim_{r\to 0} \int_{C'_2} \frac{1}{z^3-z^2+z-1} dz = \int_\pi^{2\pi} \frac{i}{(1+i)(1-i)} d\theta = i\frac{\pi}{2}$
$\mathcal{P}\int_{-\infty}^{\infty} \frac{1}{x^3-x^2+x-1} dx = \oint_{C'} \frac{1}{z^3-z^2+z-1} dz - \int_{C_1} \frac{1}{z^3-z^2+z-1} dz - \int_{C'_2} \frac{1}{z^3-z^2+z-1} dz$
$= 2\pi i \left.\frac{1}{(z-1)(z+i)}\right|_{z=i} + 2\pi i \left.\frac{1}{(z-i)(z+i)}\right|_{z=1} - i\frac{\pi}{2} = \frac{\pi(-i-1)}{2} + i\pi - i\frac{\pi}{2} = -\frac{\pi}{2}$