1. Test - Lösungen **2.12.2016**

1 Rechenbeispiele [30 Punkte, 6 Punkte je Frage]

a) $\partial_i(1/\sqrt{x_j x_j}) = -\frac{1}{2(x_k x_k)^{3/2}}\partial_i x_j x_j = -\frac{1}{2|\mathbf{x}|^3}(\delta_{ij}x_j + x_j\delta_{ij}) = -\frac{x_i}{|\mathbf{x}|^3}$

b)

$\delta_{ii}\delta_{jj} - \delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ii}\delta_{jj} - \delta_{ii} = d^2 - d$

c) $\mathbf{e}^1 = C_1(2\ -1), \quad \mathbf{e}^1 \cdot \mathbf{e}_1 = C_1 = 1 \to C_1 = 1 \to \mathbf{e}^1 = (2\ \ -1)$

$\mathbf{e}^2 = C_2(3\ -2), \quad \mathbf{e}^2 \cdot \mathbf{e}_2 = -C_2 = 1 \to C_2 = -1 \to \mathbf{e}^2 = (-3\ 2)$

alternative Lösung: $\begin{pmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

d) Da $\mathbf{g} = (g_{ij})$ und $\mathbf{g}^{-1} = (g^{ij})$, gilt $a^i{}_j = g^{ik}g_{kj} = \delta^i{}_j$. ($\mathbf{A} = (a^i{}_j)$ ist eine Einheitsmatrix.)

$\varepsilon_{ijk}a^1{}_i a^2{}_j a^3{}_k = \det(\mathbf{A}) = 1$

e) $\oint_C \frac{2z}{4z+1}dz = 2\pi i\ \frac{2z}{4}\Big|_{z=-1/4} = -\frac{1}{4}\pi i$

2 Spektraltheorem [35 Punkte]

a) Eigenwertgleichung: $\mathbf{A}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix} \to (\mathbf{A} - \lambda\mathbf{I})\begin{pmatrix} a \\ b \end{pmatrix} = 0 \to \det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$\to (1/3 - \lambda)(-1/6 - \lambda) - 1/9 = 0 \to \lambda_1 = -1/3$ und $\lambda_2 = 1/2$

b) Wenn $\lambda_1 = -1/3$, $2a + b = 0. \to \mathbf{v}_1 = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} \to \mathbf{E}_1 = \frac{\mathbf{v}_1 \otimes \mathbf{v}_1}{|\mathbf{v}_1|^2} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$

Wenn $\lambda_2 = 1/2$, $a - 2b = 0. \to \mathbf{v}_2 = c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} \to \mathbf{E}_2 = \frac{\mathbf{v}_2 \otimes \mathbf{v}_2}{|\mathbf{v}_2|^2} = \frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

Alternative Lösung :

$p_1(x) = \frac{x - \lambda_2}{\lambda_1 - \lambda_2} = -\frac{6}{5}(x - 1/2) = -\frac{1}{5}(6x - 3) \to \mathbf{E}_1 = p_1(\mathbf{T}) = -\frac{1}{5}\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$

$p_2(x) = \frac{x - \lambda_1}{\lambda_2 - \lambda_1} = \frac{6}{5}(x + 1/3) = \frac{1}{5}(6x + 2) \to \mathbf{E}_2 = p_2(\mathbf{T}) = \frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

c) $\mathbf{A} = \lambda_1\mathbf{E}_1 + \lambda_2\mathbf{E}_2 \to \mathbf{B} = \exp(\mathbf{A}) = e^{-1/3}\mathbf{E}_1 + e^{1/2}\mathbf{E}_2 = \frac{1}{5}\begin{pmatrix} e^{-1/3} + 4e^{1/2} & -2e^{-1/3} + 2e^{1/2} \\ -2e^{-1/3} + 2e^{1/2} & 4e^{-1/3} + e^{1/2} \end{pmatrix}$

d) $[\mathbf{A}, \mathbf{B}] = [\lambda_1\mathbf{E}_1 + \lambda_2\mathbf{E}_2, e^{\lambda_1}\mathbf{E}_1 + e^{\lambda_2}\mathbf{E}_2] = \lambda_1 e^{\lambda_1}\underbrace{[\mathbf{E}_1, \mathbf{E}_1]}_{=0} + \lambda_1 e^{\lambda_2}[\mathbf{E}_1, \mathbf{E}_2] + \lambda_2 e^{\lambda_1}[\mathbf{E}_2, \mathbf{E}_1] + \lambda_2 e^{\lambda_2}\underbrace{[\mathbf{E}_2, \mathbf{E}_2]}_{=0}$

$= \lambda_1 e^{\lambda_2}[\mathbf{E}_1, \mathbf{E}_2] + \lambda_2 e^{\lambda_1}[\mathbf{E}_2, \mathbf{E}_1]$

$\mathbf{v}_1$ und $\mathbf{v_2}$ sind orthogonal (oder $\mathbf{E}_1\mathbf{E}_2 = \mathbf{E}_2\mathbf{E}_1 = 0$) $\to [\mathbf{E}_1, \mathbf{E}_2] = 0 \to [\mathbf{A}, \mathbf{B}] = 0$

Alternative Lösung

$\mathbf{AB} = \frac{1}{6}\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\frac{1}{5}\begin{pmatrix} e^{-1/3} + 4e^{1/2} & -2e^{-1/3} + 2e^{1/2} \\ -2e^{-1/3} + 2e^{1/2} & 4e^{-1/3} + e^{1/2} \end{pmatrix} = \frac{1}{30}\begin{pmatrix} -2e^{-1/3} + 12e^{1/2} & 4e^{-1/3} + 6e^{1/2} \\ 4e^{-1/3} + 6e^{1/2} & -8e^{-1/3} + 3e^{1/2} \end{pmatrix}$

$\mathbf{BA} = \frac{1}{5}\begin{pmatrix} e^{-1/3} + 4e^{1/2} & -2e^{-1/3} + 2e^{1/2} \\ -2e^{-1/3} + 2e^{1/2} & 4e^{-1/3} + e^{1/2} \end{pmatrix}\frac{1}{6}\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} = \frac{1}{30}\begin{pmatrix} -2e^{-1/3} + 12e^{1/2} & 4e^{-1/3} + 6e^{1/2} \\ 4e^{-1/3} + 6e^{1/2} & -8e^{-1/3} + 3e^{1/2} \end{pmatrix}$

$\to [\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA} = 0$

3 Lokale Transformation [35 Punkte]

a)

Kartesische Basis: $\mathbf{e}_1 = \hat{\mathbf{x}}$, $\mathbf{e}_2 = \hat{\mathbf{y}}$.

Polarkoordinaten: $(x'^1, x'^2) = (r, \theta)$

Transformation der Koordinaten: $x^1 = r\cos\theta$, $x^2 = r\sin\theta$.

Transformation der Basis: $\mathbf{dx} = dx^i\mathbf{e}_i = dx'^j(\partial'_j x^i)\mathbf{e}_i = dx'^j\mathbf{e}'_j \to \mathbf{e}'_j = \partial'_j x^i\mathbf{e}_i \quad (\partial'_j x^i = \frac{\partial x^i}{\partial x'^j})$

$\mathbf{S} = (s^i{}_j) = (\partial'_j x^i) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

b)

$\mathbf{g}' = \mathbf{S}^T\mathbf{S}$ (oder $g'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j$) $\rightarrow \mathbf{g}' = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ und $\mathbf{g}'^* = \mathbf{g}'^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$

Alternative Lösung 1 für $\mathbf{g}'^*$:

$\begin{pmatrix} \mathbf{e}'^1 \\ \mathbf{e}'^2 \end{pmatrix} = \mathbf{S}^{-1} \begin{pmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -(1/r)\sin\theta & (1/r)\cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \end{pmatrix}$

$g'^{ij} = \mathbf{e}'^i \cdot \mathbf{e}'^j \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$

c) $dr/dt = 0$, $d\theta/dt = 4\pi t$

$L = \oint_{C_1} ds = \oint_{C_1} \sqrt{dr^2 + r^2 d\theta^2} = \int_0^1 \sqrt{a^2(4\pi t)^2 dt^2} = 4\pi a \int_0^1 t dt = 2\pi a$

d) Vektor auf der Kurve $C_2 : \mathbf{r} = 3\cos\theta \mathbf{e}_1 + 3\sin\theta \mathbf{e}_2$

$d\mathbf{s} = \partial_\theta \mathbf{r}\ d\theta = (-3\sin\theta\mathbf{e}_1 + 3\cos\theta\mathbf{e}_2)\ d\theta = \mathbf{e}'_2|_{r=3}\ d\theta = 9\mathbf{e}'^2|_{r=3}\ d\theta$

$\oint_{C_2} \mathbf{w} \cdot d\mathbf{s} = \int_0^{2\pi} (\sin^2\theta\mathbf{e}'_1 - (1/9)\cos^2\theta\mathbf{e}'_2) \cdot 9\mathbf{e}'^2 d\theta = -\int_0^{2\pi} \cos^2\theta\ d\theta = -\pi$

Alternative Lösung:

$\mathbf{w} = \sin^2\theta\mathbf{e}'_1 - r^{-2}\cos^2\theta\mathbf{e}'_2 = (\sin^2\theta\cos\theta + r^{-1}\cos^2\theta\sin\theta)\mathbf{e}_1 + (\sin^3\theta - r^{-1}\cos^3\theta)\mathbf{e}_2$

$\rightarrow \mathbf{w} \cdot d\mathbf{s} = -3\sin^3\theta\cos\theta - \cos^2\theta\sin^2\theta + 3\sin^3\theta\cos\theta - \cos^4\theta = -\cos^2\theta \rightarrow \oint_{C_2} \mathbf{w} \cdot d\mathbf{s} = -\pi$