2. Test - Lösungen **19.1.2018**

1 Rechenbeispiele [30 Punkte, 6 Punkte je Frage]

a) $\int_0^\infty (\delta(x-1)+\delta(x+1))e^x dx = e$
b) $\int_{-\infty}^\infty\int_{-\infty}^\infty \delta(E-x^2-y^2)dxdy = \frac{d}{dE}\int_{-\infty}^\infty\int_{-\infty}^\infty H(E-x^2-y^2)dxdy = \frac{d}{dE}(\pi E) = \pi$
Alternative Lösung 1:
$\int_{-\infty}^\infty\int_{-\infty}^\infty \delta(E-x^2-y^2)dxdy = \int_0^\infty dr \int_0^{2\pi} d\theta\ r\delta(E-r^2) = 2\pi\int_0^\infty dr\, r\,\delta(E-r^2)$
$= \begin{cases} 2\pi\int_0^\infty dt\delta(E-t)\frac{1}{2} = \pi \\ 2\pi\int_0^\infty dr\, r\,\frac{1}{2\sqrt{E}}(\delta(r-\sqrt{E})+\delta(r+\sqrt{E})) = \pi \end{cases}$
Alternative Lösung 2:
Wenn $E > x^2$, $\delta(E-x^2-y^2) = \frac{1}{2\sqrt{E-x^2}}(\delta(y-\sqrt{E-x^2})+\delta(y+\sqrt{E-x^2}))$
$\int_{-\infty}^\infty\int_{-\infty}^\infty \delta(E-x^2-y^2)dxdy = \int_{-\sqrt{E}}^{\sqrt{E}} dx\frac{1}{2\sqrt{E-x^2}}\int_{-\infty}^\infty dy(\delta(y-\sqrt{E-x^2})+\delta(y+\sqrt{E-x^2})) = \int_{-\sqrt{E}}^{\sqrt{E}} dx\frac{1}{\sqrt{E-x^2}}$
$x = \sqrt{E}\sin t \to dx = \sqrt{E}\cos t dt$
$\int_{-\sqrt{E}}^{\sqrt{E}} dx\frac{1}{\sqrt{E-x^2}} = \int_{-\pi/2}^{\pi/2} dt\sqrt{E}\cos t\frac{1}{\sqrt{E(1-sin^2 t)}} = \int_{-\pi/2}^{\pi/2} dt\cos t\frac{1}{\cos t} = \frac{1}{\sqrt{E}}\pi$
c) $\int_0^\infty f'(x)\sin x dx = f(x)\sin x|_{x=0}^\infty - \int_0^\infty f(x)\cos x dx = -\int_0^{\pi/2}\sin x\cos x dx = -\frac{1}{2}\int_0^{\pi/2}\sin 2x dx = -\frac{1}{2}$
Alternative Lösung
$f(x) = H(\pi/2+x)H(\pi/2-x)\sin x$
$\to f'(x) = H(\pi/2+x)H(\pi/2-x)\cos x + \delta(\pi/2+x)H(\pi/2-x)\sin x - H(\pi/2+x)\delta(\pi/2-x)\sin x$
$= H(\pi/2+x)H(\pi/2-x)\cos x - \delta(\pi/2+x)H(\pi/2-x) - H(\pi/2+x)\delta(\pi/2-x)$
$\int_0^\infty f'(x)\sin x dx$
$= \int_0^\infty H(\pi/2+x)H(\pi/2-x)\cos x\sin x dx - \int_0^\infty \delta(\pi/2+x)H(\pi/2-x)\sin x dx - \int_0^\infty H(\pi/2+x)\delta(\pi/2-x)\sin x dx$
$= \int_0^{\pi/2}\cos x\sin x dx - \int_0^{\pi/2}\delta(\pi/2+x)\sin x dx - \int_0^\infty \delta(\pi/2-x)\sin x dx = 1/2 + 0 - 1 = -1/2$
d) $t = x^2 \to dx = 1/(2\sqrt{t})dt$
$\int_0^\infty x^3 e^{-x^2}dx = \int_0^\infty t^{3/2}e^{-t}\frac{1}{2\sqrt{t}}dt = \frac{1}{2}\int_0^\infty te^{-t}dt = \frac{1}{2}\Gamma(2) = \frac{1}{2}$
Alternative Lösung: $-\frac{1}{2}\frac{d}{dx}(x^2e^{-x^2}) = x^3e^{-x^2} - xe^{-x^2}$
$\to \int_0^\infty x^3e^{-x^2}dx = -\frac{1}{2}x^2e^{-x^2}\Big|_{x=0}^\infty + \int_0^\infty xe^{-x^2}dx = \int_0^\infty t^{1/2}e^{-t}\frac{1}{2\sqrt{t}}dt = \frac{1}{2}\int_0^\infty e^{-t}dt = \frac{1}{2}$
e) $t = \sin^2\theta \to dt = 2\sin\theta\cos\theta d\theta = 2t^{1/2}(1-t)^{1/2}d\theta$
$\int_0^{\pi/2}\sin^4\theta d\theta = \int_0^1 t^2\frac{1}{2}t^{-1/2}(1-t)^{-1/2}dt = \frac{1}{2}\int_0^1 t^{3/2}(1-t)^{-1/2}dt = \frac{1}{2}B(5/2,1/2) = \frac{1}{2}\Gamma(5/2)\Gamma(1/2)/\Gamma(3) = \frac{1}{2}\frac{3}{4}\sqrt{\pi}\sqrt{\pi}/2 = \frac{3}{16}\pi$
Alternative Lösung:
$\sin^4\theta = \frac{1}{4}(1-\cos(2\theta))^2 = \frac{1}{4}-\frac{1}{2}\cos(2\theta)+\frac{1}{4}\cos^2(2\theta) = \frac{1}{4}-\frac{1}{2}\cos(2\theta)+\frac{1}{8}(1+\cos(4\theta)) = \frac{3}{8}-\frac{1}{2}\cos(2\theta))+\frac{1}{8}\cos(4\theta)$
$\int_0^{\pi/2}\sin^4\theta d\theta = \frac{3}{8}\int_0^{\pi/2}d\theta - \frac{1}{2}\int_0^{\pi/2}\cos(2\theta))d\theta + \frac{1}{8}\int_0^{\pi/2}\cos(4\theta))d\theta = \frac{3}{16}\pi$

2 Greensche Funktion [40 Punkte]

a) $\left(\frac{d}{dt}+1-i\right)\left(\frac{d}{dt}-1-i\right)G_I = \delta(t-t')$
Ansatz : $G_I(t,t') = (2\pi)^{-1}\int_{-\infty}^\infty \tilde{G}_I(\omega)\,e^{i\omega(t-t')}d\omega$
Inhomogene Gleichung: $\mathcal{L}_t G_I(t,t') = \delta(t-t') \to \frac{1}{2\pi}\int_{-\infty}^\infty (i\omega+1-i)(i\omega-1-i)\tilde{G}_I(\omega)\,e^{i\omega(t-t')}d\omega = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega(t-t')}d\omega$
Vergleich der Integranden: $\tilde{G}_I(\omega) = \frac{1}{(i\omega+1-i)(i\omega-1-i)} = -\frac{1}{(\omega-1-i)(\omega-1+i)}$
Fourier-Transformation $G_I(t,t') = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega = -\frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{i\omega(t-t')}}{(\omega-1-i)(\omega-1+i)}d\omega$
$= -\frac{1}{2\pi}H(t-t')\lim_{R\to\infty}\left(\oint_{C_1}\frac{e^{i\omega(t-t')}}{(\omega-1-i)(\omega-1+i)}d\omega - \int_{c_1}\frac{e^{i\omega(t-t')}}{(\omega-1-i)(\omega-1+i)}d\omega\right)$
$+\frac{1}{2\pi}H(t'-t)\lim_{R\to\infty}\left(\oint_{C_2}\frac{e^{i\omega(t-t')}}{(\omega-1-i)(\omega-1+i)}d\omega - \int_{c_2}\frac{e^{i\omega(t-t')}}{(\omega-1-i)(\omega-1+i)}d\omega\right)$
wobei C_1 (C_2) der obere (untere) geschloßene Halbkreis mit Radius R ist und c_1 (c_2) der obere (untere) offene Halbkreis.

Im Limes $R \to \infty$ konvergiert die Integrale (2. und 4. Integrale) entlang des oberen offenen Halbkreises gegen null. Nach dem Residuensatz können die restlichen Integrale berechnet werden:
$G_I(t,t') = -H(t-t')\frac{1}{2}e^{(-1+i)(t-t')} - H(t'-t)\frac{1}{2}e^{(1+i)(t-t')}$
b)
$\left(\frac{d}{dt} - 1 - i\right) G_I = -\delta(t-t')\frac{1}{2} + \delta(t'-t)\frac{1}{2}$
$-H(t-t')\frac{1}{2}(-1+i)e^{(-1+i)(t-t')} - H(t'-t)\frac{1}{2}(1+i)e^{(1+i)(t-t')} - (1+i)G_I(t,t')$
$= -H(t-t')\frac{1}{2}(-1+i)e^{(-1+i)(t-t')} + H(t-t')\frac{1}{2}(1+i)e^{(-1+i)(t-t')} = H(t-t')e^{(-1+i)(t-t')}$
$\left(\frac{d}{dt} + 1 - i\right)\left(\frac{d}{dt} - 1 - i\right) G_I = \left(\frac{d}{dt} + 1 - i\right) H(t-t')e^{(-1+i)(t-t')}$
$= \delta(t-t') + (-1+i)H(t-t')e^{(-1+i)(t-t')} + (1-i)H(t-t')e^{(-1+i)(t-t')} = \delta(t-t')$
Alternative Lösung:
$\left(\frac{d}{dt} + 1 - i\right)\left(\frac{d}{dt} - 1 - i\right) G_I = \frac{d^2G_I}{dt^2} - 2i\frac{dG_I}{dt} - 2G_I$
$\frac{d}{dt}G_I(t,t') = -\delta(t-t')\frac{1}{2}e^{(-1+i)(t-t')} + \delta(t'-t)\frac{1}{2}e^{(1+i)(t-t')}$
$-H(t-t')\frac{1}{2}(-1+i)e^{(-1+i)(t-t')} - H(t'-t)\frac{1}{2}(1+i)e^{(1+i)(t-t')}$
$= -\delta(t-t')\frac{1}{2} + \delta(t'-t)\frac{1}{2} - H(t-t')\frac{1}{2}(-1+i)e^{(-1+i)(t-t')} - H(t'-t)\frac{1}{2}(1+i)e^{(1+i)(t-t')}$
$= -H(t-t')\frac{1}{2}(-1+i)e^{(-1+i)(t-t')} - H(t'-t)\frac{1}{2}(1+i)e^{(1+i)(t-t')}$
$\frac{d^2}{dt^2}G_I(t,t') = -\delta(t-t')\frac{1}{2}(-1+i) + \delta(t'-t)\frac{1}{2}(1+i)$
$-H(t-t')\frac{1}{2}(-1+i)^2e^{(-1+i)(t-t')} - H(t'-t)\frac{1}{2}(1+i)^2e^{(1+i)(t-t')}$
$= \delta(t-t') + iH(t-t')e^{(-1+i)(t-t')} - iH(t'-t)e^{(1+i)(t-t')}$
$\frac{d^2}{dt^2}G_I(t,t') - 2i\frac{d}{dt}G_I(t,t') = \delta(t-t') + iH(t-t')e^{(-1+i)(t-t')} - iH(t'-t)e^{(1+i)(t-t')}$
$+2iH(t-t')\frac{1}{2}(-1+i)e^{(-1+i)(t-t')} + 2iH(t'-t)\frac{1}{2}(1+i)e^{(1+i)(t-t')}$
$= \delta(t-t') - H(t-t')e^{(-1+i)(t-t')} - H(t'-t)e^{(1+i)(t-t')} = \delta(t-t') + 2G_I(t,t')$
$\to$ erfüllt die imhomogene Gleichung.
c) $y_I(t) = \int_0^\infty G_I(t,t')\delta(t'-T)dt' = -\int_0^t \frac{1}{2}e^{(-1+i)(t-t')}\delta(t'-T)dt' - \int_t^\infty \frac{1}{2}e^{(1+i)(t-t')}\delta(t'-T)dt'$
Wenn $t < T$, $y_I(t) = -\frac{1}{2}e^{(1+i)(t-T)}$, $y_I'(t) = -\frac{1}{2}(1+i)e^{(1+i)(t-T)}$
Wenn $t > T$, $y_I(t) = -\frac{1}{2}e^{(-1+i)(t-T)}$, $y_I'(t) = -\frac{1}{2}(-1+i)e^{(-1+i)(t-T)}$
$y_I(0) = -\frac{1}{2}e^{-(1+i)T} \neq 0$ $(t = 0 < T)$
$y_I'(0) = -\frac{1}{2}(1+i)e^{-(1+i)T} \neq 0$
homogene Greensche Funktion : $\left(\frac{d}{dt} + 1 - i\right)\left(\frac{d}{dt} - 1 - i\right) G_0 = 0$
$\to G_1(t,t') = e^{(-1+i)(t-t')}$ oder $G_2(t,t') = e^{(1+i)(t-t')}$
$y_1(t) = \int_0^\infty e^{(-1+i)(t-t')}\delta(t'-T)dt' = e^{(-1+i)(t-T)}$
$y_2(t) = \int_0^\infty e^{(1+i)(t-t')}\delta(t'-T)dt' = e^{(1+i)(t-T)}$.
$y(t) = y_I(t) + A_1y_1(t) + A_2y_2(t) \to y(0) = -\frac{1}{2}e^{-(1+i)T} + A_1e^{(1-i)T} + A_2e^{-(1+i)T}$
$y'(0) = -\frac{1}{2}(1+i)e^{-(1+i)T} + A_1(-1+i)e^{(-1+i)T} + A_2(1+i)e^{-(1+i)T}$
Wenn $A_1 = 0$ und $A_2 = 1/2$, $y(0) = y'(0) = 0$.
Lösung: $y(t) = -\frac{1}{2}e^{(1+i)(t-T)} + \frac{1}{2}e^{(1+i)(t-T)}$ für $t < T$, und $y(t) = -\frac{1}{2}e^{(-1+i)(t-T)} + \frac{1}{2}e^{(1+i)(t-T)}$ für $t > T$

3 Differentialgleichung [30 Punkte]

a) Die Sturm-Liouville'schen Gestalt:
$\left(\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x) + \lambda\rho(x)\right)y(x) = 0 \to p(x)y''(x) + p'(x)y'(x) + q(x)y(x) + \lambda\rho(x)y(x) = 0$
$4x^2y'' + 8xy' + (4x + 1 + 4\lambda x^2)y = 0$
Vergleich der Koeffizienten :
$p'(x)/p(x) = 2x/x^2 \to \log p(x) = 2\log x \to p(x) = x^2$
$(q(x) + \lambda\rho(x))/p(x) = (x + 1/4 + \lambda x^2)/x^2$
$\to q(x) + \lambda\rho(x) = x + 1/4 + \lambda x^2 \to q(x) = x + 1/4$ und $\rho(x) = x^2$
$\to \left(\frac{d}{dx}\left[x^2\frac{d}{dx}\right] + x + \frac{1}{4} + \lambda x^2\right)y(x) = 0$
b) Wenn $p(x) = \rho(x)$, gilt $t(x) = x$. D.h. die Differentialgleichungen in der Sturm-Liouville'schen Gestalt und in der Liouville'schen Normalform haben die gleiche Variable.
Ansatz: $w(x) = u(x)y(x)$
Differentialgleichungen in der Liouville'schen Normalform: $-uy'' - 2u'y' + (-u'' + vu - \lambda u)y = 0$
Vergleich der Koeffizienten : $2u'/u = 2x/x^2 \to \log u = \log x \to u = x$
$-(-u'' + vu - \lambda u)/u = (x + 1/4 + \lambda x^2)/x^2 \to -v + \lambda = (x + 1/4)/x^2 + \lambda \to v(x) = -(x + 1/4)/x^2$
$-w''(x) + \left(-\frac{x+1/4}{x^2} + \lambda\right)w(x) = 0$

c) Ansatz : $\sum_{n=0}^{\infty} a_n x^{n+\sigma}$ mit $a_0 \neq 0$

$-w''(x) + \left(-\frac{x+1/4}{x^2} + \lambda\right) w(x)$

$= -\sum_{n=0}^{\infty} a_n(n+\sigma)(n+\sigma-1)x^{n+\sigma-2} - \sum_{n=0}^{\infty} a_n x^{n+\sigma-1} - \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{n+\sigma-2} + \lambda \sum_{n=0}^{\infty} a_n x^{n+\sigma}$

$= -\sum_{n=0}^{\infty} a_n(n+\sigma)(n+\sigma-1)x^{n+\sigma-2} - \sum_{n=1}^{\infty} a_{n-1} x^{n+\sigma-2} - \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{n+\sigma-2} + \lambda \sum_{n=2}^{\infty} a_{n-2} x^{n+\sigma-2} = 0$

Die Gleichung gilt für beliebige x. $\rightarrow$ Alle Koeffizienten der $x^{\sigma+n}$-Term müssen null sein.

$x^{\sigma-2}$-Term: $(-\sigma(\sigma-1) - 1/4)a_0 = 0 \rightarrow$ wegen $a_0 \neq 0$, $\sigma = 1/2$

$x^{\sigma-1}$-Term: $-a_1\sigma(1+\sigma) - a_0 - \frac{1}{4}a_1 - a_1 - a_0 = 0 \rightarrow a_1 = -a_0$

$x^{n+\sigma-2}$-Term $(n \geq 2)$: $-a_n(n+\sigma)(n+\sigma-1) - a_{n-1} - (1/4)a_n + \lambda a_{n-2} = 0 \rightarrow -a_n n^2 - a_{n-1} + \lambda a_{n-2} = 0$

$\rightarrow a_n = -\frac{1}{n^2}a_{n-1} + \frac{\lambda}{n^2}a_{n-2}$

$a_2 = \frac{1}{4}(\lambda a_0 - a_1) = \frac{\lambda+1}{4}a_0$

$a_3 = \frac{1}{9}(\lambda a_1 - a_2) = \frac{1}{9}(-\lambda a_0 - (1/4)(\lambda+1)a_0) = -\frac{5\lambda+1}{36}a_0$

$\left(a_4 = \frac{1}{16}(\lambda a_2 - a_3) = \frac{1}{16}((1/4)\lambda(\lambda+1)a_0 + (1/36)(5\lambda+1)a_0) = \frac{9\lambda^2+14\lambda+1}{576}a_0\right)$

$\rightarrow w(x) = x^{1/2} - x^{3/2} + \frac{\lambda+1}{4}x^{5/2} - \frac{5\lambda+1}{36}x^{7/2} + \cdots$

$\rightarrow y(x) = w(x)/x = x^{-1/2} - x^{1/2} + \frac{\lambda+1}{4}x^{3/2} - \frac{5\lambda+1}{36}x^{5/2} + \cdots$

Anmerkung : Ursprünglich war es vorgesehen, dass die Gleichung für $\lambda = 0$ gelöst werden soll. Aber in der Angabe war die Bedingung $\lambda = 0$ fehlend. Deshalb werden die meisten Punkte gegeben wenn die Rekursionsgleichung richtig abgeleitet ist.