4. Tutorium - Lösungen **10.11.2017**

- ANMERKUNG: Es liegt in der Verantwortung des Einzelnen, sich die Beispiele zunächst alleine und ganz ohne Hilfsmittel anzuschauen. Google, Wolfram Alpha, Lösungssammlungen, etc. helfen nur kurzfristig - leider nicht beim Test!

4.1 Differentialoperatoren

a) $\text{grad}|\mathbf{x}| = \nabla|\mathbf{x}| \to \partial_j \sqrt{x_i x_i} = \frac{x_j}{|\mathbf{x}|}$ (siehe Bsp.2.1f).

b) $\text{div rot}\mathbf{v} \to \partial_i \varepsilon_{ijk} \partial_j v_k = \underbrace{\varepsilon_{ijk}}_{\text{antisymmetrisch}} \underbrace{\partial_i \partial_j}_{\text{symmetrisch}} v_k = 0$

c) $\text{rot}(\mathbf{x}/|\mathbf{x}|) \to \varepsilon_{ijk} \partial_j \frac{x_k}{\sqrt{x_l x_l}} = \varepsilon_{ijk} \left(\frac{1}{\sqrt{x_l x_l}} \delta_{jk} - \frac{2 x_k x_l}{2(x_m x_m)\sqrt{x_n x_n}} \delta_{jl} \right) = -\varepsilon_{ijk} \frac{x_k x_j}{(x_m x_m)\sqrt{x_n x_n}} = 0$

d) $\text{div}(\mathbf{x} \times \mathbf{p}) \to \partial_i (\mathbf{x} \times \mathbf{p})_i = \partial_i \varepsilon_{ijk} x_j p_k = \varepsilon_{ijk} \delta_{ij} p_k = 0$

e) $\text{rot}(\mathbf{F} \times \mathbf{x}) \to \varepsilon_{ijk} \partial_j \varepsilon_{klm} F_l x_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial_j F_l x_m = \partial_j (F_i x_j) - \partial_j (F_j x_i)$
$= (\partial_j F_i) x_j + \delta_{jj} F_i - (\partial_j F_j) x_i - F_j \delta_{ji} = (x_j \partial_j) F_i + 3F_i - (\partial_j F_j) x_i - F_i \to (\mathbf{x} \cdot \nabla)\mathbf{F} - \mathbf{x}(\nabla \cdot \mathbf{F}) + 2\mathbf{F}$

4.2 Spektraltheorem und Kommutator

a) Eigenwerte : $(\lambda + 3)(\lambda - 3) - 16 = 0 \to \lambda_1 = 5,\ \lambda_2 = -5$

b) Die Polynome, $p_1(t) = \frac{t - \lambda_2}{\lambda_1 - \lambda_2} = \frac{1}{10}(t+5)$ und $p_2(t) = \frac{t - \lambda_1}{\lambda_2 - \lambda_1} = -\frac{1}{10}(t-5)$, erfüllen $p_i(\lambda_j) = \delta_{ij}$.

c) $p_i(\mathbf{A}) = \sum_j p_i(\lambda_j) \mathbf{E}_j = \mathbf{E}_i$

d) $\mathbf{E}_1 = p_1(\mathbf{A}) = \frac{1}{10} \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ und $\mathbf{E}_2 = p_2(\mathbf{A}) = -\frac{1}{10} \begin{pmatrix} -8 & -4 \\ -4 & -2 \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}$

$\mathbf{A} = \lambda_1 p_1(\mathbf{A}) + \lambda_2 p_2(\mathbf{A}) = 5\frac{1}{5} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} - 5\frac{1}{5} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

e) Eigenwerte : $(\lambda + 5)(\lambda - 10) - 100 = 0 \to \lambda_1 = 15,\ \lambda_2 = -10$

$\mathbf{E_1} = p_1(\mathbf{B}) = \frac{\mathbf{B} - \lambda_2 \mathbf{I}}{\lambda_1 - \lambda_2} = \frac{1}{25} \begin{pmatrix} 5 & -10 \\ -10 & 20 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$

$\mathbf{E_2} = p_2(\mathbf{B}) = \frac{\mathbf{B} - \lambda_1 \mathbf{I}}{\lambda_2 - \lambda_1} = -\frac{1}{25} \begin{pmatrix} -20 & -10 \\ -10 & -5 \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}$

$\mathbf{B} = 15\frac{1}{5} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} - 10\frac{1}{25} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

f) Weil die Projektoren $\mathbf{E}_i$ aus (a) und (b) sind identisch, gilt:
$[\mathbf{A}, \mathbf{B}] = [(5\mathbf{E}_1 - 5\mathbf{E}_2), (15\mathbf{E}_1 - 10\mathbf{E}_2)] = (-5 \times 15)[\mathbf{E}_2, \mathbf{E}_1] + (-5 \times 10)[\mathbf{E}_1, \mathbf{E}_2] = 0$

4.3 Metrischer Tensor

a) $V = \mathbf{f}_1 \cdot \mathbf{f}_2 \times \mathbf{f}_3 = 2$

$\mathbf{f}^1 = \frac{1}{V} \mathbf{f}_2 \times \mathbf{f}_3 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{f}^2 = \frac{1}{V} \mathbf{f}_3 \times \mathbf{f}_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{f}^3 = \frac{1}{V} \mathbf{f}_1 \times \mathbf{f}_2 = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

b) $\mathbf{x} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \end{pmatrix} \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} \to \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{pmatrix}}_{=\mathbf{T}} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$

$\to \mathbf{T} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$

c) $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \end{pmatrix} = \begin{pmatrix} x'_1 & x'_2 & x'_3 \end{pmatrix} \begin{pmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{pmatrix}$

$\rightarrow \begin{pmatrix} x'_1 & x'_2 & x'_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \end{pmatrix}}_{=\mathbf{T}^*} \rightarrow \mathbf{T}^* = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \mathbf{T}^{-1} \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \mathbf{T}^* \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}$ und $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = ((\mathbf{T}^*)^{-1})^T \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \mathbf{T}^T \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}$

Da gilt $x^i = x_i$, $\quad \mathbf{T}^* \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \mathbf{T}^T \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \rightarrow \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = (\mathbf{T}^T)^{-1}\mathbf{T}^* \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \mathbf{T}^{*T}\mathbf{T}^* \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}$

$\rightarrow \mathbf{T}^{*T}\mathbf{T}^* = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$\mathbf{T}^{*T}\mathbf{T}^* = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} \begin{pmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \end{pmatrix} = (\mathbf{f}_i \cdot \mathbf{f}_j) = \mathbf{g}'$

e) $\sqrt{x_i x^i} = \sqrt{1+1+1} = \sqrt{3}$

$\begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x'_1 & x'_2 & x'_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \end{pmatrix}$

$\sqrt{x'_i x'^i} = \sqrt{1+1+1} = \sqrt{3}$

$(g'_{ij} x'^j) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

$\sqrt{x'^i g_{ij} x'^j} = \sqrt{(1/2) \times 2 + (1/2) \times 2 + (1/2) \times 2} = \sqrt{3}$

f) $V = (\mathbf{f}_1 \times \mathbf{f}_2) \cdot \mathbf{f}_3 = \det(\mathbf{T}^*)$

$\mathbf{g}' = (\mathbf{T}^*)^T \mathbf{T}^* \rightarrow \det(\mathbf{g}') = \det((\mathbf{T}^*)^T)\det(\mathbf{T}^*) = [\det(\mathbf{T}^*)]^2 = V^2$