5. Tutorium - Lösungen **17.11.2017**

- ANMERKUNG: Es liegt in der Verantwortung des Einzelnen, sich die Beispiele zunächst alleine und ganz ohne Hilfsmittel anzuschauen. Google, Wolfram Alpha, Lösungssammlungen, etc. helfen nur kurzfristig - leider nicht beim Test!

5.1 Tensoren

a) $\mathbf{S} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

b) $A'_{ij} = s^k{}_i A_{kl} s^l{}_j \rightarrow \mathbf{S}^T (A_{kl}) \mathbf{S}$

$\rightarrow \begin{pmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$

c) $(g'_{ij}) = (\mathbf{f}_i \cdot \mathbf{f}_j) = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \quad (g'^{ij}) = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$

d)

$A'^{ij}(A'_{ji} - A'_{ij}) = (\mathbf{S}^{-1})^i{}_k A^{kl} (\mathbf{S}^{-1})^j{}_l (s^m{}_j A_{mn} s^n{}_i - s^m{}_i A_{mn} s^n{}_j) = A^{kl} A_{mn} \delta^m{}_l \delta^n{}_k - A^{kl} A_{mn} \delta^m{}_k \delta^n{}_l$

$= A^{kl} A_{lk} - A^{kl} A_{kl} = A^{kl}(A_{lk} - A_{kl})$

Alternative Lösung: $(A'^{ij}) = \mathbf{g}^{-1} (A'_{ij}) \mathbf{g}^{-1} = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$

$\rightarrow (A'^{ij}(A'_{ji} - A'_{ij})) = \operatorname{tr} \left[\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] = \operatorname{tr} \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} = -1$

$A^{kl}(A_{lk} - A_{kl}) \rightarrow \operatorname{tr} \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] = -1$

5.2 Lokale Transformation

a) $\mathbf{x} = x^i \mathbf{e}_i$ mit $\mathbf{e}_1 = \hat{\mathbf{x}}$, $\mathbf{e}_2 = \hat{\mathbf{y}}$ und $\mathbf{e}_3 = \hat{\mathbf{z}}$.

Infinitesimale Änderung von $\mathbf{x}$: $\mathbf{x} + \mathbf{dx} = \mathbf{x} + dx^i \mathbf{e}_i$.

Lokale Basistransformation (linearisierte Basistransformation für die infinitesimalen Änderung):

$$\mathbf{x} + \mathbf{dx} = \mathbf{x} + dx^i \mathbf{e}_i = \mathbf{x} + (\tfrac{\partial}{\partial x'^j} x^i) dx'^j \mathbf{e}_i \equiv \mathbf{x} + dx'^j \mathbf{e}'_j. \rightarrow \mathbf{e}'_j = (\tfrac{\partial}{\partial x'^j} x^i) \mathbf{e}_i$$

Zylinderkoordinaten : $x'^1 = \rho$, $x'^2 = \theta$, $x'^3 = z$

Transformation zwischen kartesischen Koordinaten und Zylinderkoordinaten

$x^1 = r \cos\theta$, $x^2 = r \sin\theta$ und $x^3 = z$.

$\mathbf{e}'_1 = (\frac{\partial}{\partial x'^1} x^i) \mathbf{e}_i = \cos\theta \, \mathbf{e}_1 + \sin\theta \, \mathbf{e}_2$

$\mathbf{e}'_2 = (\frac{\partial}{\partial x'^2} x^i) \mathbf{e}_i = -\rho \sin\theta \, \mathbf{e}_1 + \rho \cos\theta \, \mathbf{e}_2$

$\mathbf{e}'_3 = (\frac{\partial}{\partial x'^3} x^i) \mathbf{e}_i = \mathbf{e}_3$

$$\begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} \mathbf{S} \text{ mit } s^i{}_j = \frac{\partial}{\partial x'^j} x^i$$

b) $\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j} = s^j{}_i \frac{\partial}{\partial x^j}$

Die Ableitung transformiert genau wie die Basisvektoren mit $\mathbf{S}$. $\rightarrow \frac{\partial}{\partial x'^i}$ formt ein kovarianter Vektor und wird mit unten stehenden Index ($\frac{\partial}{\partial x'^i} = \partial'_i$) geschrieben.

c)

$$(g'_{ij}) = (\mathbf{e}'_i \cdot \mathbf{e}'_j) = \mathbf{S}^T \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (g'^{ij}) = (g'_{ij})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

d) $ds = \sqrt{dx^i dx_i} = \sqrt{dx^i g_{ij} dx^j} = \sqrt{(\partial'_k x^i) dx'^k \delta_{ij} (\partial'_l x^j) dx'^l} = \underbrace{\sqrt{s^i{}_k s^i{}_l dx'^k dx'^l}}_{\mathbf{g}=\mathbf{S}^T\mathbf{S}} = \sqrt{g'_{kl} dx'^k dx'^l}$

$= \sqrt{d\rho^2 + \rho^2 d\theta^2 + dz^2}$
e) Länge der Helix C : $\int_C ds = \int_0^n dt\sqrt{4\pi^2\rho_0^2 + h^2} = n\sqrt{4\pi^2\rho_0^2 + h^2}$
f) Vektor auf dem Zylinder : $\mathbf{x} = x^i(x'^2 = \theta, x'^3 = z)\,\mathbf{e}_i$ ($\rho = \rho_0$: Konstante)
$d\mathbf{F} = (\partial_2'\mathbf{x}) \times (\partial_3'\mathbf{x})dx'^2dx'^3 \to (\partial_2' x^i\mathbf{e}_i) \times (\partial_3' x^j\mathbf{e}_j)dx'^2dx'^3 = \mathbf{e}_2' \times \mathbf{e}_3' dx'^2dx'^3 = (\rho_0\cos\theta\mathbf{e}_1 + \rho_0\sin\theta\mathbf{e}_2)d\theta dz$
$= \rho_0\mathbf{e}_1' d\theta dz$
Alternative Lösung:
Vektor auf dem Zylinder : $\mathbf{x} = x\,\mathbf{e}_1 + \sqrt{\rho_0^2 - x^2}\mathbf{e}_2 + z\mathbf{e}_3$
$d\mathbf{F} = (\partial_z\mathbf{x}) \times (\partial_x\mathbf{x})dxdz = (-\frac{x}{\sqrt{\rho_0^2 - x^2}}\,\mathbf{e}_1 - \mathbf{e}_2)dxdz = (-\frac{1}{\tan\theta}\,\mathbf{e}_1 - \mathbf{e}_2)| - \rho_0\sin\theta|d\theta dz = (\cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2)\rho_0 d\theta dz$
$= \mathbf{e}_1'\rho_0 d\theta dz$
g) $dV = dxdydz = \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,z)}\right| d\rho\, d\theta\, dz = |\det(\partial_j' x^i)| d\rho\, d\theta\, dz = |\det(\mathbf{S})| d\rho\, d\theta\, dz$
$= \sqrt{\det(\mathbf{g}')} d\rho\, d\theta\, dz = \rho d\rho\, d\theta\, dz$
Anmerkung : $dV = ((\mathbf{e}_1 \times \mathbf{e}_2)\cdot\mathbf{e}_3)dx^1dx^2dx^3 = ((\mathbf{e}_1' \times \mathbf{e}_2')\cdot\mathbf{e}_3')dx'^1dx'^2dx'^3$

5.3 Gaußsche und Stokessche Integralsatz

a) $\mathbf{w} = y\mathbf{e}_x + x\mathbf{e}_y + z^2\mathbf{e}_z \to \text{div}\mathbf{w} = \partial_x y + \partial_y x + \partial_z z^2 = 2z$
$\int_V \text{div}\mathbf{w} dv = \int_0^1 dz \int_{x^2+y^2<z} dxdy 2z = \underbrace{\int_0^1 dz \int_0^{\sqrt{z}} \rho d\rho \int_0^{2\pi}}_{\text{Bsp.5.2g}} d\theta\, 2z = 4\pi \int_0^1 zdz \int_0^{\sqrt{z}} \rho d\rho = 2\pi \int_0^1 z^2 dz = \frac{2}{3}\pi$

Anmerkung :
In Zylinderkoordinaten, $\mathbf{w} = y\mathbf{e}_x + x\mathbf{e}_y + z^2\mathbf{e}_z = 2\rho\sin\theta\cos\theta\mathbf{e}_1' + (\cos^2\theta - \sin^2\theta)\mathbf{e}_2' + z^2\mathbf{e}_3'$
$\partial_1'\mathbf{e}_1' = 0,\ \ \partial_2'\mathbf{e}_1' = (1/\rho)\mathbf{e}_2',\ \ \partial_3'\mathbf{e}_1' = 0,$
$\partial_1'\mathbf{e}_2' = (1/\rho)\mathbf{e}_2',\ \ \partial_2'\mathbf{e}_2' = -\rho\mathbf{e}_1',\ \ \partial_3'\mathbf{e}_2' = 0,$
$\partial_1'\mathbf{e}_3' = 0,\ \ \partial_2'\mathbf{e}_3' = 0,\ \ \partial_3'\mathbf{e}_3' = 0,$
$\to \mathbf{e}'^1\cdot\partial_1'\mathbf{w} = 2\sin\theta\cos\theta\underbrace{\mathbf{e}'^1\cdot\mathbf{e}_1'}_{=1} + 2\rho\sin\theta\cos\theta\mathbf{e}'^1\cdot\underbrace{\partial_1'\mathbf{e}_1'}_{=0} + (\cos^2\theta - \sin^2\theta)\underbrace{\mathbf{e}'^1\cdot\partial_1'\mathbf{e}_2'}_{=0} + z^2\mathbf{e}'^1\cdot\underbrace{\partial_1'\mathbf{e}_3'}_{=0} = 2\sin\theta\cos\theta$
$\mathbf{e}'^2\cdot\partial_2'\mathbf{w} = 2\rho\sin\theta\cos\theta\mathbf{e}'^2\cdot\partial_2'\mathbf{e}_1' - 4\cos\theta\sin\theta\mathbf{e}'^2\cdot\mathbf{e}_2' = -2\sin\theta\cos\theta$
$\mathbf{e}'^3\cdot\partial_3'\mathbf{w} = 2z\mathbf{e}'^3\cdot\mathbf{e}_3' = 2z$
$\to \text{div}\mathbf{w} = 2\sin\theta\cos\theta - 2\sin\theta\cos\theta + 2z = 2z$

b) Vektor auf der Oberfläche F_1 : $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + \mathbf{e}_z \to$ Tangentialvektoren : $\partial_x\mathbf{r} = \mathbf{e}_x$ und $\partial_y\mathbf{r} = \mathbf{e}_y$
Oberflächenelement : $d\mathbf{f}_1 = \partial_x\mathbf{r}dx \times \partial_y\mathbf{r}dy = \mathbf{e}_z dxdy \to \int_{F_1}\mathbf{w}|_{z=1}\cdot d\mathbf{f}_1 = \int_{\rho<1}\rho d\rho d\theta = \pi$

In Zylinderkoordinaten : F_1 : $\mathbf{r}|_{z=1} = \rho\mathbf{e}_1' + \mathbf{e}_3' \to$ Tangentialvektoren : $\partial_\rho\mathbf{r} = \mathbf{e}_1'$ und $\partial_\theta\mathbf{r} = \rho\partial_2'\mathbf{e}_1' = \mathbf{e}_2'$
Oberflächenelement : $d\mathbf{f}_1 = \partial_\rho\mathbf{r} \times \partial_\theta\mathbf{r}d\rho d\theta = \mathbf{e}_1' \times \mathbf{e}_2' d\rho d\theta = \rho\mathbf{e}'^3 d\rho d\theta \to \int_{F_1}\mathbf{w}|_{z=1}\cdot d\mathbf{f}_1 = \int_{\rho<1}\rho d\rho d\theta = \pi$

c) Vektor auf der Oberfläche F_2 : $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + (x^2 + y^2)\mathbf{e}_z$
$\to$ Tangentialvektoren : $\partial_x\mathbf{r} = \mathbf{e}_x + 2x\mathbf{e}_z$ und $\partial_y\mathbf{r} = \mathbf{e}_y + 2y\mathbf{e}_z$
Oberflächenelement : $d\mathbf{f}_2 = -\partial_x\mathbf{r}dx \times \partial_y\mathbf{r}dxdy = (2x\mathbf{e}_x + 2y\mathbf{e}_y - \mathbf{e}_z)dxdy$
(Anmerkung : Das Vorzeichen des Flächenvektors wird bestimmt, mit, z.B. $d\mathbf{f}_2 = -\mathbf{e}_z$ für $x = y = 0$).
$\to \int_{F_2}\mathbf{w}|_{z=x^2+y^2}\cdot d\mathbf{f}_2 = \int_{F_2}(4xy - (x^2+y^2)^2)dxdy = \int_0^1 \rho d\rho \int_0^{2\pi} d\theta(4\rho^2\sin\theta\cos\theta - \rho^4) = -2\pi\int_0^1 \rho^5 d\rho = -\frac{1}{3}\pi$
$\int_F \mathbf{w}\cdot d\mathbf{f} = \int_{F_1}\mathbf{w}\cdot d\mathbf{f}_1 + \int_{F_2}\mathbf{w}\cdot d\mathbf{f}_2 = \frac{2}{3}\pi$

In Zylinderkoordinaten: F_2 : $\mathbf{r}|_{z=\rho^2} = \rho\mathbf{e}_1' + \rho^2\mathbf{e}_3' \to$ Tangentialvektoren : $\partial_\rho\mathbf{r} = \mathbf{e}_1' + 2\rho\mathbf{e}_3'$ und $\partial_\theta\mathbf{r} = \mathbf{e}_2'$
Oberflächenelement : $d\mathbf{f}_2 = -\partial_\rho\mathbf{r}dx \times \partial_\theta\mathbf{r}d\rho d\theta = (2\rho^2\,\mathbf{e}'^1 - \rho\mathbf{e}'^3)d\rho d\theta$
$\to \int_{F_2}\mathbf{w}|_{z=\rho^2}\cdot d\mathbf{f}_2 = \int_{F_2}(2\rho^3\sin\theta\cos\theta - \rho^5)d\rho d\theta = -\frac{1}{3}\pi$

d) Vektor auf der Oberfläche F : $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + (5 - x^2 - y^2)\mathbf{e}_z$
$\to$ Tangentialvektoren : $\partial_x\mathbf{r} = \mathbf{e}_x - 2x\mathbf{e}_z$ und $\partial_y\mathbf{r} = \mathbf{e}_y - 2y\mathbf{e}_z$
Oberflächenelement : $d\mathbf{f} = \partial_x\mathbf{r}dx \times \partial_y\mathbf{r}dy = (2x\mathbf{e}_x + 2y\mathbf{e}_y + \mathbf{e}_z)dxdy$
$\mathbf{b} = z^2\mathbf{e}_x + 4xy^2\mathbf{e}_y + xy\mathbf{e}_z \to \text{rot}\mathbf{b} = x\mathbf{e}_x + (2z - y)\mathbf{e}_y + 4y^2\mathbf{e}_z$
$\int_F \text{rot}\mathbf{b}|_{z=5-x^2-y^2}\cdot d\mathbf{f} = \int_F dxdy(2x^2 + 2y(10 - 2x^2 - 2y^2 - y) + 4y^2) = \int_F dxdy2(x^2 + y^2)$

$= \int_0^2 \rho d\rho \int_0^{2\pi} d\theta 2\rho^2 = 4\pi \int_0^2 \rho^3 d\rho = 16\pi$

In Zylinderkoordinaten, $F : \mathbf{r} = \rho \mathbf{e}'_1 + (5-\rho^2)\mathbf{e}'_3 \rightarrow$ Tangentialvektoren : $\partial_\rho \mathbf{r} = \mathbf{e}'_1 - 2\rho \mathbf{e}'_3$ und $\partial_\theta \mathbf{r} = \mathbf{e}'_2$
Oberflächenelement : $d\mathbf{f} = \partial_\rho \mathbf{r} d\rho \times \partial_\theta \mathbf{r} d\theta = (-2\rho^2 \mathbf{e}'^1 + \rho \mathbf{e}'^3) d\rho d\theta$
$\mathrm{rot}\mathbf{b} = (\rho \cos(2\theta) - 2z \sin\theta)\mathbf{e}'_1 + (2z\rho^{-1} - \sin\theta)\cos\theta \mathbf{e}'_2 + 4\rho^2 \sin^2\theta \mathbf{e}'_3$
$\rightarrow \int_F \mathrm{rot}\mathbf{b}|_{z=5-x^2-y^2} \cdot d\mathbf{f} = \int_F (-2\rho^3 \cos(2\theta) + 4(5-\rho^2)\rho^2 \sin\theta + 4\rho^3 \sin^2\theta) d\rho d\theta = 4 \int_F \rho^3 \sin^2\theta d\rho d\theta = 16\pi$

e) $C = \{(x,y,z)|x^2+y^2=4, z=1\} \rightarrow d\mathbf{s} = \partial_\theta(2\cos\theta \mathbf{e}_x + 2\sin\theta \mathbf{e}_y)d\theta = (-2\sin\theta \mathbf{e}_x + 2\cos\theta \mathbf{e}_y)d\theta$
$\mathbf{b}|_C = \mathbf{e}_x + 32\cos\theta \sin\theta^2 \mathbf{e}_y + 4\cos\theta \sin\theta \mathbf{e}_z$
$\oint_C \mathbf{b} \cdot d\mathbf{s} = \int_0^{2\pi} d\theta(-2\sin\theta + 64\cos^2\theta \sin\theta^2) = 16 \int_0^{2\pi} d\theta \sin^2(2\theta) = 8 \int_0^{2\pi} d\theta(1 - \cos(4\theta)) = 16\pi$

In Zylinderkoordinaten, $C = \{(\rho,\theta,z)|\rho = 2, 0 \leq \theta < 2\pi, z = 1\}$
$\rightarrow \mathbf{r} = 2\mathbf{e}'_1 + \mathbf{e}_3 \rightarrow d\mathbf{s} = \partial_\theta \mathbf{r} d\theta = \mathbf{e}'_2 d\theta = 4\mathbf{e}'^2 d\theta$
$\mathbf{b}|_C = (\cos\theta + 32\cos\theta \sin^3\theta)\mathbf{e}'_1 + (-(1/2)\sin\theta + 16\cos^2\theta \sin^2\theta)\mathbf{e}'_2 + 4\cos\theta \sin\theta \mathbf{e}'_3$
$\oint_C \mathbf{b} \cdot d\mathbf{s} = \int_0^{2\pi} d\theta(-2\sin\theta + 64\cos^2\theta \sin^2\theta) = 16\pi$