6. Tutorium - Lösungen **24.11.2017**

- ANMERKUNG: Es liegt in der Verantwortung des Einzelnen, sich die Beispiele zunächst alleine und ganz ohne Hilfsmittel anzuschauen. Google, Wolfram Alpha, Lösungssammlungen, etc. helfen nur kurzfristig - leider nicht beim Test!

6.1 Residuensatz

a) $\oint_C \frac{z}{2z^2+4z-6}dz = \oint_C \frac{z}{2(z+3)(z-1)}dz = 2\pi i \left(\frac{z}{2(z-1)}\Big|_{z=-3} + \frac{z}{2(z+3)}\Big|_{z=1} \right) = 2\pi i \left(\frac{3}{8}+\frac{1}{8}\right) = \pi i$

b) $\oint_C \frac{1}{2z^2-3z-2}dz = \oint_C \frac{1}{2(z+1/2)(z-2)}dz = 2\pi i \; \frac{1}{2(z-2)}\Big|_{z=-1/2} = 2\pi i \left(-\frac{1}{5}\right) = -\frac{2}{5}\pi i$

c) Wenn $R > 2$,

$\oint_C \frac{1}{2z^2-3z-2}dz = \oint_C \frac{1}{2(z+1/2)(z-2)}dz = 2\pi i \; \frac{1}{2(z-2)}\Big|_{z=-1/2} + 2\pi i \; \frac{1}{2(z+1/2)}\Big|_{z=2} = 2\pi i \left(-\frac{1}{5}+\frac{1}{5}\right) = 0$

oder

$\left|\oint_C \frac{1}{2z^2-3z-2}dz\right| = \left|\int_0^{2\pi} \frac{1}{2R^2e^{2i\theta}-3Re^{i\theta}-2} iRe^{i\theta}d\theta\right| \le \int_0^{2\pi}\left|\frac{1}{2R^2e^{2i\theta}-3Re^{i\theta}-2} iRe^{i\theta}\right| d\theta = \int_0^{2\pi}\left|\frac{R}{2R^2e^{2i\theta}-3Re^{i\theta}-2}\right| d\theta$

$\xrightarrow{R\to\infty} \int_0^{2\pi} \left|\frac{1}{2R}\right| d\theta \to 0, \qquad \oint_C \frac{1}{2z^2-3z-2}dz = 0$

d) $\oint_C \frac{z^3-z^2+z+1}{2(z-1)^3}dz = \oint_C \frac{1}{2(z-1)^3}\left[2+2(z-1)+2(z-1)^2+(z-1)^3\right]dz$

$= \oint_C \left[\frac{1}{(z-1)^3}+\frac{1}{(z-1)^2}+\frac{1}{z-1}+\frac{1}{2}\right] = 2\pi i$

oder $\oint_C \frac{z^3-z^2+z+1}{2(z-1)^3}dz = 2\pi i \frac{1}{2!}\frac{d^2}{dz^2}\frac{1}{2}(z^3-z^2+z+1)\Big|_{z=1} = 2\pi i \frac{1}{2!}\frac{1}{2}(6z-2)\big|_{z=1} = 2\pi i$

e) $\oint_C \frac{e^{itz}}{z^2+1}dz = \oint_C \frac{e^{itz}}{(z+i)(z-i)}dz = 2\pi i \; \frac{e^{itz}}{z+i}\Big|_{z=i} = \pi e^{-t}$

$\left|\int_{C_1} \frac{e^{itz}}{z^2+1}dz\right| = \left|\int_0^{2\pi} \frac{e^{itR\exp i\theta}}{R^2e^{2i\theta}+1} iRe^{i\theta}d\theta\right| \le \int_0^{2\pi}\left|\frac{e^{itR\exp i\theta}}{R^2e^{2i\theta}+1}iRe^{i\theta}\right| d\theta = \int_0^{2\pi} \frac{R|e^{itR\exp i\theta}|}{|R^2e^{2i\theta}+1|}d\theta$

Da $|e^{itR\exp i\theta}| = |e^{itRcos\theta}||e^{-tR\sin\theta}| = |e^{-tR\sin\theta}| \xrightarrow{R\to\infty} 0$ wenn $t > 0$ und $\sin\theta > 0$ (d.h. $0 < \theta < \pi$)

und $\frac{R}{|R^2e^{2i\theta}+1|} \xrightarrow{R\to\infty} 0$, $\int_0^{2\pi} \frac{R|e^{itR\exp i\theta}|}{|R^2e^{2i\theta}+1|}d\theta \xrightarrow{R\to\infty} 0$

$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}dx = \lim_{R\to\infty}\left(\oint_C \frac{e^{itz}}{z^2+1}dz - \int_{C_1}\frac{e^{itz}}{z^2+1}dz\right) = \pi e^{-t}$

6.2 Differentialoperatoren in krummlinigen Koordinaten

a) $\nabla x^i = \mathbf{f}^j \partial_j x^i = \mathbf{f}^j {\delta_j}^i = \mathbf{f}^i$

b) $\nabla \times (\nabla x^i) \to \underbrace{\varepsilon_{jkl}}_{\text{asymmetrisch}} \; \underbrace{\partial_k\partial_l}_{\text{symmetrisch}} \; x^i = 0$ (siehe Bsp.4.1b)

c) $V = \mathbf{f}_1 \cdot (\mathbf{f}_2 \times \mathbf{f}_3) = \sqrt{\det(\mathbf{g})}$ (siehe Bsp.4.3f) $\to \mathbf{f}^1 = \frac{1}{V}\mathbf{f}_2 \times \mathbf{f}_3$ und $\mathbf{f}_1 = V\,\mathbf{f}^2 \times \mathbf{f}^3$ (siehe Bsp.3.2d)

$\nabla\cdot(\frac{1}{V}\mathbf{f}_1) = \nabla\cdot(\mathbf{f}^2\times\mathbf{f}^3) = (\nabla\times \underbrace{\mathbf{f}^2}_{=\nabla x^2 \text{ (Bsp.a)}})\cdot\mathbf{f}^3 - \mathbf{f}^2\cdot(\nabla\times\underbrace{\mathbf{f}^3}_{=\nabla x^3}) = (\underbrace{\nabla\times\nabla x^2}_{=0 \text{ (Bsp.b)}})\cdot\mathbf{f}^3 - \mathbf{f}^2\cdot(\underbrace{\nabla\times\nabla x^3}_{=0}) = 0$

Analog für $\nabla\cdot(\frac{1}{V}\mathbf{f}_2) = 0$ und $\nabla\cdot(\frac{1}{V}\mathbf{f}_3) = 0$.

d) $\mathbf{v} = v^i\mathbf{f}_i$

$\nabla\cdot\mathbf{v} = \mathbf{f}^j\cdot\partial_j\left(v^i\mathbf{f}_i\right) = \mathbf{f}^j\cdot\partial_j\left(Vv^i\frac{1}{V}\mathbf{f}_i\right) = \mathbf{f}^j\cdot\partial_j\left(Vv^i\right)\frac{1}{V}\mathbf{f}_i + Vv^i\underbrace{\mathbf{f}^j\cdot\partial_j\left(\frac{1}{V}\mathbf{f}_i\right)}_{=0\text{(Bsp.b)}} = \frac{1}{V}\partial_j\left(Vv^i\right){\delta^j}_i = \frac{1}{V}\partial_i\left(Vv^i\right)$

e) $\mathbf{v} = \nabla\psi(\mathbf{x}) = \mathbf{f}_i\partial^i\psi(\mathbf{x})$

$\nabla^2\psi(\mathbf{x}) = \frac{1}{V}\partial_i\left(V\partial^i\psi(\mathbf{x})\right) = \frac{1}{V}\partial_i\left(Vg^{ij}\partial_j\psi(\mathbf{x})\right)$

f) Für orthogonale Koordinaten ist der metrische Tensor diagonal. $\to V = \sqrt{\det(\mathbf{g})} = \sqrt{g_{11}g_{22}g_{33}}$

$\nabla^2\psi(\mathbf{x}) = \frac{1}{\sqrt{g_{11}g_{22}g_{33}}}\sum_{i=1}^3\partial_i\left(\sqrt{g_{11}g_{22}g_{33}}g^{ii}\partial_i\psi(\mathbf{x})\right) = \frac{1}{\sqrt{g_{11}g_{22}g_{33}}}\sum_{i=1}^3\partial_i\left(\sqrt{g_{11}g_{22}g_{33}}(g_{ii})^{-1}\partial_i\psi(\mathbf{x})\right)$

6.3 Orthogonale krummlinige Koordinaten

a) Kugelkoordinaten: $x'^1 = r$, $x'^2 = \theta$, $x'^3 = \phi$
$\mathbf{e}'_1 = \partial'_1 x^i \mathbf{e}_i = \sin\theta\cos\phi\,\mathbf{e}_1 + \sin\theta\sin\phi\,\mathbf{e}_2 + \cos\theta\,\mathbf{e}_3$
$\mathbf{e}'_2 = \partial'_2 x^i \mathbf{e}_i = r\cos\theta\cos\phi\,\mathbf{e}_1 + r\cos\theta\sin\phi\,\mathbf{e}_2 - r\sin\theta\,\mathbf{e}_3$
$\mathbf{e}'_3 = \partial'_3 x^i \mathbf{e}_i = -r\sin\theta\sin\phi\,\mathbf{e}_1 + r\sin\theta\cos\phi\,\mathbf{e}_2$

$$\rightarrow \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} \mathbf{S} \text{ mit } \mathbf{S} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & \end{pmatrix}$$

b) $$(g'_{ij}) = (\mathbf{e}'_i \cdot \mathbf{e}'_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

c) $\sqrt{g_{11}g_{22}g_{33}} = r^2\sin\theta$
$\nabla^2\psi(\mathbf{x}) = \frac{1}{r^2\sin\theta}\left(\partial_r\left(r^2\sin\theta\partial_r\psi(\mathbf{x})\right) + \partial_\theta\left(\sin\theta\partial_\theta\psi(\mathbf{x})\right) + \partial_\phi\left((\sin\theta)^{-1}\partial_\phi\psi(\mathbf{x})\right)\right)$
$= \frac{1}{r^2}\partial_r\left(r^2\partial_r\psi(\mathbf{x})\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\psi(\mathbf{x})\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\psi(\mathbf{x})$

d) $\sqrt{g_{11}g_{22}g_{33}} = \rho$
$\nabla^2\psi(\mathbf{x}) = \frac{1}{\rho}\left(\partial_\rho\left(\rho\partial_\rho\psi(\mathbf{x})\right) + \partial_\theta\left(\rho^{-1}\partial_\theta\psi(\mathbf{x})\right) + \partial_z\left(\rho\partial_z\psi(\mathbf{x})\right)\right)$
$= \frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho\psi(\mathbf{x})\right) + \frac{1}{\rho^2}\partial_\theta^2\psi(\mathbf{x}) + \partial_z^2\psi(\mathbf{x})$

e) $\mathbf{e}'_1 = (\frac{\partial}{\partial x'^1}x^i)\mathbf{e}_i = v\cos\theta\,\mathbf{e}_1 + v\sin\theta\,\mathbf{e}_2 + u\,\mathbf{e}_3$
$\mathbf{e}'_2 = (\frac{\partial}{\partial x'^2}x^i)\mathbf{e}_i = u\cos\theta\,\mathbf{e}_1 + u\sin\theta\,\mathbf{e}_2 - v\,\mathbf{e}_3$
$\mathbf{e}'_3 = (\frac{\partial}{\partial x'^3}x^i)\mathbf{e}_i = -uv\sin\theta\,\mathbf{e}_1 + uv\cos\theta\,\mathbf{e}_2$

$$\begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} \begin{pmatrix} v\cos\theta & u\cos\theta & -uv\sin\theta \\ v\sin\theta & u\sin\theta & uv\cos\theta \\ u & -v & 0 \end{pmatrix} \equiv \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} \mathbf{S}$$

f) $$(g'_{ij}) = (\mathbf{e}'_i \cdot \mathbf{e}'_j) = \begin{pmatrix} u^2+v^2 & 0 & 0 \\ 0 & u^2+v^2 & 0 \\ 0 & 0 & u^2v^2 \end{pmatrix}$$

g) $\sqrt{g_{11}g_{22}g_{33}} = (u^2+v^2)uv$
$\nabla^2\psi(\mathbf{x}) = \frac{1}{(u^2+v^2)uv}\left(\partial_u\left(uv\partial_u\psi(\mathbf{x})\right) + \partial_v\left(uv\partial_v\psi(\mathbf{x})\right) + \partial_\theta\left(\frac{u^2+v^2}{uv}\partial_\theta\psi(\mathbf{x})\right)\right)$
$= \frac{1}{(u^2+v^2)u}\partial_u\left(u\partial_u\psi(\mathbf{x})\right) + \frac{1}{(u^2+v^2)v}\partial_v\left(v\partial_v\psi(\mathbf{x})\right) + \frac{1}{u^2v^2}\partial_\theta^2\psi(\mathbf{x})$