7. Tutorium - Lösungen **15.12.2017**

- ANMERKUNG: Es liegt in der Verantwortung des Einzelnen, sich die Beispiele zunächst alleine und ganz ohne Hilfsmittel anzuschauen. Google, Wolfram Alpha, Lösungssammlungen, etc. helfen nur kurzfristig - leider nicht beim Test!

7.1 Delta-Distribution und Heaviside-Funktion

a) $t = 2x \to dx = (1/2)dt$
$\int_0^\infty \delta(2x-\pi)\,(\sin(3x)+2)\,dx = \frac{1}{2}\int_0^\infty \delta(t-\pi)\,(\sin(3t/2)+2)\,dt = \frac{1}{2}(\sin(3\pi/2)+2) = \frac{1}{2}$
oder $\int_0^\infty \delta(2x-\pi)\,(\sin(3x)+2)\,dx = \int_0^\infty \frac{1}{2}\delta(x-\pi/2)\,(\sin(3x)+2)\,dx = \frac{1}{2}$
b) $t = x^2 - 1 \to$ Wenn $-\infty \le x \le 0$, $\infty \ge t \ge -1$ und wenn $0 \le x \le \infty$, $-1 \le t \le \infty$.
$\int_{-\infty}^\infty \delta(x^2-1)\,e^x\,dx = \int_{-\infty}^0 \delta(x^2-1)\,e^x\,dx + \int_0^\infty \delta(x^2-1)\,e^x\,dx$
$= \int_\infty^{-1} \delta(t)\,e^{-\sqrt{t+1}}\left(-\frac{1}{2\sqrt{t+1}}\right)dt + \int_{-1}^\infty \delta(t)\,e^{\sqrt{t+1}}\,\frac{1}{2\sqrt{t+1}}dt = \int_{-1}^\infty \delta(t)\,e^{-\sqrt{t+1}}\,\frac{1}{2\sqrt{t+1}}dt + \int_{-1}^\infty \delta(t)\,e^{\sqrt{t+1}}\,\frac{1}{2\sqrt{t+1}}dt$
$= \frac{1}{2}(e^{-1}+e)$
oder $\int_{-\infty}^\infty \delta(x^2-1)\,e^x\,dx = \int_{-\infty}^\infty \frac{1}{2}(\delta(x+1)+\delta(x-1))\,e^x\,dx = \frac{1}{2}(e^{-1}+e)$
c) $y = 2z \to dy = 2dz$

$$\int_{-\infty}^\infty\int_{-\infty}^\infty H\left(C - x^2 - \frac{y^2}{4}\right)dxdy = 2\underbrace{\int_{-\infty}^\infty\int_{-\infty}^\infty H\left(C-x^2-z^2\right)dxdz}_{\text{Fläche eines Kreises mit Radius}\sqrt{C}} = 2\pi C$$

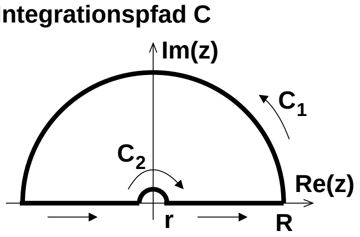
oder $\int_{-\infty}^\infty\int_{-\infty}^\infty H\left(C - x^2 - \frac{y^2}{4}\right)dxdy = 2\int_{-\infty}^\infty\int_{-\infty}^\infty H\left(C-x^2-z^2\right)dxdz$
Polarkoordinaten $x = r\cos\theta$, $z = r\sin\theta$. Für $x^2+z^2 = r^2 < C$, $0 \le r \le \sqrt{r}$ und $0 \le \theta \le 2\pi$.
$2\int_{-\infty}^\infty\int_{-\infty}^\infty H\left(C-x^2-z^2\right)dxdz = 2\int_0^{2\pi} d\theta \int_0^{\sqrt{C}} rdr = 2\pi C$
d) $\int_{-\infty}^\infty\int_{-\infty}^\infty \delta\left(C-x^2-\frac{y^2}{4}\right)dxdy = \int_{-\infty}^\infty\int_{-\infty}^\infty \frac{d}{dC}H\left(C-x^2-\frac{y^2}{4}\right)dxdy$
$= \frac{d}{dC}\int_{-\infty}^\infty\int_{-\infty}^\infty H\left(C-x^2-\frac{y^2}{4}\right)dxdy = \frac{d}{dC}(2\pi C) = 2\pi$
oder
$\int_{-\infty}^\infty\int_{-\infty}^\infty \delta\left(C-x^2-\frac{y^2}{4}\right)dxdy = 2\int_{-\infty}^\infty\int_{-\infty}^\infty \delta\left(C-x^2-z^2\right)dxdz = 2\int_0^{2\pi}\int_0^{\sqrt{C}} \delta\left(C-r^2\right)rdrd\theta$
$= 4\pi\int_0^\infty \delta\left(C-r^2\right)rdr = 4\pi\frac{1}{2\sqrt{C}}\sqrt{C} = 2\pi$

7.2 Verallgemeinerte Funktion

a) $\frac{d}{dx}e^{|x|} = \frac{d}{dx}(H(-x)e^{-x}+H(x)e^x) = -\delta(-x)e^{-x} - H(-x)e^{-x} + \delta(x)e^x + H(x)e^x = -\delta(x) - H(-x)e^{-x} + \delta(x) + H(x)e^x = -H(-x)e^{-x} + H(x)e^x$
$\frac{d^2}{dx^2}e^{|x|} = \frac{d}{dx}(-H(-x)e^{-x}+H(x)e^x) = \delta(-x)e^{-x} + H(-x)e^{-x} + \delta(x)e^x + H(x)e^x = \delta(x) + H(-x)e^{-x} + \delta(x) + H(x)e^x = 2\delta(x) + e^{|x|}$
b) $f(x) = H(x)H(\pi-x)\cos x$
$\frac{d}{dx}f(x) = \delta(x)H(\pi-x)\cos x - H(x)\delta(\pi-x)\cos x - H(x)H(\pi-x)\sin x = \delta(x)+\delta(\pi-x) - H(x)H(\pi-x)\sin x$
$\int_{\pi/2}^\infty f'(x)\cos x dx = \int_{\pi/2}^\infty(\delta(x)+\delta(\pi-x)-H(x)H(\pi-x)\sin x)\cos x dx = -1 - \int_{\pi/2}^\pi \sin x\cos x dx = -1+\frac{1}{2} = -\frac{1}{2}$
Alternative Lösung für das Integral:
$\int_{\pi/2}^\infty f'(x)\cos x dx = f(x)\cos x|_{x=\pi/2}^\infty + \int_{\pi/2}^\infty f(x)\sin x dx = \int_{\pi/2}^\pi \cos x\sin x dx = -\frac{1}{2}$
Anmerkung: $\int_{\pi/2}^\infty f'(x)\cos x dx \ne \int_{\pi/2}^\pi(-\sin x)\cos x dx = \frac{1}{2}$

7.3 Cauchyscher Hauptwert

a) Wenn $0 \le \theta \le \pi/2$, $\sin\theta \ge \frac{2}{\pi}\theta \to -R\sin\theta \le -\frac{2R}{\pi}\theta \to e^{-R\sin\theta} \le e^{-\frac{2R}{\pi}\theta}$
$\int_0^{\pi/2} e^{-R\sin\theta} d\theta \le \int_0^{\pi/2} e^{-\frac{2R}{\pi}\theta} d\theta = \frac{\pi}{2R}(1-e^{-R}) < \frac{\pi}{2R}$



b) $\int_{C_1} \frac{e^{iz}}{z} dz = \int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta} d\theta = \int_0^\pi ie^{iRe^{i\theta}} d\theta$
$= \int_0^\pi ie^{iR\cos\theta - R\sin\theta} d\theta \le \int_0^\pi \left|ie^{iR\cos\theta - R\sin\theta}\right| d\theta = \int_0^\pi e^{-R\sin\theta} d\theta$
$= 2\int_0^{\pi/2} e^{-R\sin\theta} d\theta < \frac{\pi}{R} \xrightarrow{R\to\infty} 0$

c) $\int_{C_2} \frac{e^{iz}}{z} dz = \int_\pi^0 \frac{e^{ire^{i\theta}}}{re^{i\theta}} ire^{i\theta} d\theta = i\int_\pi^0 e^{ire^{i\theta}} d\theta \xrightarrow{r\to 0} i\int_\pi^0 d\theta = -i\pi$

d) $\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \oint_C \frac{e^{iz}}{z} dz - \lim_{R\to\infty}\int_{C_1} \frac{e^{iz}}{z} dz - \lim_{r\to 0}\int_{C_2} \frac{e^{iz}}{z} dz = 0 - 0 + i\pi$

e) $\int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty \frac{e^{ix}-e^{-ix}}{2ix} dx = -\frac{i}{2}\int_0^\infty \frac{e^{ix}}{x} dx + \underbrace{\frac{i}{2}\int_0^\infty \frac{e^{-ix}}{x} dx}_{x\to -x} = -\frac{i}{2}\int_0^\infty \frac{e^{ix}}{x} dx - \frac{i}{2}\int_{-\infty}^0 \frac{e^{ix}}{x} dx$

$\lim_{x\to 0} \frac{\sin x}{x} = 1 \to$ keine Singularität
$\int_0^\infty \frac{\sin x}{x} dx = -\frac{i}{2}\mathcal{P}\int_{-\infty}^\infty \frac{e^{ix}}{x} dx = \frac{\pi}{2}$

7.4 Greensche Funktion

Ansatz : $G_I(t,t') = (2\pi)^{-1}\int_{-\infty}^\infty \tilde{G}_I(\omega)\, e^{i\omega(t-t')} d\omega$
Inhomogene Gleichung : $\mathcal{L}_t G_I(t,t') = \delta(t-t') \to \frac{1}{2\pi}\int_{-\infty}^\infty (i\omega+\gamma)\tilde{G}_I(\omega)\, e^{i\omega(t-t')} d\omega = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega(t-t')} d\omega$
Vergleich der Integranden: $(i\omega+\gamma)\tilde{G}_I(\omega) = 1 \to \tilde{G}_I(\omega) = \frac{1}{i\omega+\gamma}$
Fourier-Transformation $G_I(t,t') = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega = -\frac{i}{2\pi}\int_{-\infty}^\infty \frac{e^{i\omega(t-t')}}{\omega - i\gamma} d\omega$
Das Integral kann mit $\int_{-\infty}^\infty e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega = \oint_{C_1} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega - \int_{\tilde{C}_1} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega$
oder $-\oint_{C_2} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega + \int_{\tilde{C}_2} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega$ gerechnet werden.
(C_1: geschlossener Halbkreis und $\tilde{C}_1$: offener Halbkreis in der oberen komplexen Ebene, C_2: geschlossener Halbkreis und $\tilde{C}_2$: offener Halbkreis in der unteren komplexen Ebene,)
Auf den Halbkreisen glit $|e^{i\omega(t-t')}| = |\exp(iRe^{i\theta}(t-t'))| = \exp(-R\sin\theta(t-t'))$
Im Limes $R\to\infty$, $|e^{i\omega(t-t')}| \to 0$ auf dem oberen Halbkreis $\tilde{C}_1$ (d.h. $\sin\theta > 0$) wenn $t > t'$ oder auf dem unteren Halbkreis $\tilde{C}_2$ (d.h. $\sin\theta < 0$) wenn $t < t'$.

$$G_I(t,t') = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega(t-t')}\tilde{G}_I(\omega) = \frac{1}{2\pi}H(t-t')\lim_{R\to\infty}\Big[\underbrace{\oint_{C_1} e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega}_{=2\pi e^{-\gamma(t-t')}} - \underbrace{\int_{\tilde{C}_1} e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega}_{=0\ \text{(Bsp.7.3)}}\Big]$$

$$+\frac{1}{2\pi}H(t'-t)\lim_{R\to\infty}\Big[-\underbrace{\oint_{C_2} e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega}_{=0} + \underbrace{\int_{\tilde{C}_2} e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega}_{=0\ \text{(Bsp.7.3)}}\Big] = H(t-t')e^{-\gamma(t-t')}$$