1. Test - Lösungen **30.11.2018**

1 Rechenbeispiele [30 Punkte, 6 Punkte je Frage]

a) $\frac{1}{x_k x_k}\partial_i(x_i x_j x_j) = \frac{1}{x_k x_k}(\partial_i x_i)x_j x_j + 2\frac{1}{x_k x_k}x_i x_j(\partial_i x_j) = \frac{x_j x_j}{x_k x_k}\delta_{ii}x_j x_j + 2\frac{x_i x_j}{x_k x_k}\delta_{ij} = 3\frac{x_j x_j}{x_k x_k} + 2\frac{x_j x_j}{x_k x_k} = 5$

b) $\partial_i\sqrt{x_j x_j} = \frac{1}{2(x_k x_k)^{1/2}}\partial_i x_j x_j = \frac{1}{2|\mathbf{x}|}2\delta_{ij}x_j = \frac{x_i}{|\mathbf{x}|}(\text{ oder } \frac{x_i}{\sqrt{x_k x_k}})$

c) $\mathbf{e}^1\cdot\mathbf{e}_2 = 0 \to \mathbf{e}^1 = C_1(1\ -2),\quad \mathbf{e}^1\cdot\mathbf{e}_1 = -C_1 = 1 \to C_1 = -1 \to \mathbf{e}^1 = (-1\ 2)$

$\mathbf{e}^2\cdot\mathbf{e}_1 = 0 \to \mathbf{e}^2 = C_2(1\ -1),\quad \mathbf{e}^2\cdot\mathbf{e}_2 = C_2 = 1 \to \mathbf{e}^2 = (1\ -1)$

alternative Lösung: $\begin{pmatrix}\mathbf{e}^1\\ \mathbf{e}^2\end{pmatrix} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{pmatrix}^{-1} = \begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}^{-1} = \begin{pmatrix}-1 & 2\\ 1 & -1\end{pmatrix}$

d) $\varepsilon_{ijk}a^1{}_i a^2{}_j a^3{}_k = \det(a^i{}_j) = -4+1 = -3$

e) Da $\mathbf{g}^* = \mathbf{g}^{-1}$ und $g_{ij} = g_{ji}$, $g^{ij}g_{ij} = (\mathbf{g}^*\mathbf{g}^T)_{ii} = (\mathbf{g}^*\mathbf{g})_{ii} = \mathrm{Tr}(\mathbf{1}) = d$

2 Tensoren [40 Punkte]

a) Eigenwertgleichung: $\begin{pmatrix}a^{11} & a^{12}\\ a^{21} & a^{22}\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \lambda\begin{pmatrix}u\\ v\end{pmatrix} \to \begin{pmatrix}a^{11}-\lambda & a^{12}\\ a^{21} & a^{22}-\lambda\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = 0$

$\to \det\begin{pmatrix}-\lambda & 1\\ 1 & -\lambda\end{pmatrix} = \lambda^2 - 1 = 0 \to \lambda_1 = -1$ und $\lambda_2 = 1$

b) Eigenvektor: $\mathbf{e}'_i = s^j{}_i\mathbf{e}_j = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{pmatrix}\begin{pmatrix}s^1{}_i\\ s^2{}_i\end{pmatrix}$

$\begin{pmatrix}a^{11} & a^{12}\\ a^{21} & a^{22}\end{pmatrix}\begin{pmatrix}s^1{}_i\\ s^2{}_i\end{pmatrix} = \begin{pmatrix}s^2{}_i\\ s^1{}_i\end{pmatrix} = \lambda\begin{pmatrix}s^1{}_i\\ s^2{}_i\end{pmatrix} \to s^2{}_i = \lambda s^1{}_i$ und $s^1{}_i = \lambda s^2{}_i$

Wenn $\lambda_1 = -1$, $\begin{pmatrix}s^1{}_1\\ s^2{}_1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}\left(\text{oder } \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 1\end{pmatrix}\right)$. Wenn $\lambda_2 = 1$, $\begin{pmatrix}s^1{}_2\\ s^2{}_2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$.

Alternative :

a) und b)

Eigenwertgleichung $\begin{pmatrix}a^{11} & a^{12}\\ a^{21} & a^{22}\end{pmatrix}\begin{pmatrix}s^1{}_i\\ s^2{}_i\end{pmatrix} = \begin{pmatrix}s^2{}_i\\ s^1{}_i\end{pmatrix} = \lambda\begin{pmatrix}s^1{}_i\\ s^2{}_i\end{pmatrix} \to s^2{}_i = \lambda s^1{}_i$ und $s^1{}_i = \lambda s^2{}_i$

$\to s^1{}_i = \lambda^2 s^1{}_i \to \lambda^2 = 1 \to \lambda_1 = -1$ und $\lambda_2 = 1 \to \begin{pmatrix}s^1{}_1\\ s^2{}_1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$ und $\begin{pmatrix}s^1{}_2\\ s^2{}_2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$.

c) In der kartesischen Basis werden die Eigenvektoren $\mathbf{e}'_i$ mit $\begin{pmatrix}s^1{}_i\\ s^2{}_i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \mp 1\end{pmatrix}$ dargestellt.

Deshalb sind die Projektoren in der kartesischen Basis gegeben, durch

$\begin{pmatrix}P^{11} & P^{12}\\ P^{21} & P^{22}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\ -1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$

$\begin{pmatrix}Q^{11} & Q^{12}\\ Q^{21} & Q^{22}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\ 1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$

d) Transformationsmatrix : $\mathbf{S} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$

Die kontravarianten Komponenten werden mit der Inverse $\mathbf{S}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$ transformiert.

$\begin{pmatrix}P'^{11} & P'^{12}\\ P'^{21} & P'^{22}\end{pmatrix} = \mathbf{S}^{-1}\begin{pmatrix}P^{11} & P^{12}\\ P^{21} & P^{22}\end{pmatrix}(\mathbf{S}^{-1})^T = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$

$\begin{pmatrix}Q'^{11} & Q'^{12}\\ Q'^{21} & Q'^{22}\end{pmatrix} = \mathbf{S}^{-1}\begin{pmatrix}Q^{11} & Q^{12}\\ Q^{21} & Q^{22}\end{pmatrix}(\mathbf{S}^{-1})^T = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$

Alternative :

c) und d)

In der Eigenbasis sind die Projektoren $\begin{pmatrix}P'^{11} & P'^{12}\\ P'^{21} & P'^{22}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ und $\begin{pmatrix}Q'^{11} & Q'^{12}\\ Q'^{21} & Q'^{22}\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$

In der kartesischen Basis

$$\begin{pmatrix} P^{11} & P^{12} \\ P^{21} & P^{22} \end{pmatrix} = \mathbf{S} \begin{pmatrix} P'^{11} & P'^{12} \\ P'^{21} & P'^{22} \end{pmatrix} \mathbf{S}^T = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix} = \mathbf{S} \begin{pmatrix} Q'^{11} & Q'^{12} \\ Q'^{21} & Q'^{22} \end{pmatrix} \mathbf{S}^T = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

e)

$$\mathbf{A} = \lambda_1 \mathbf{E}_1 + \lambda_2 \mathbf{E}_2 \rightarrow \begin{pmatrix} a'^{11} & a'^{12} \\ a'^{21} & a'^{22} \end{pmatrix} = \lambda_1 \begin{pmatrix} P'^{11} & P'^{12} \\ P'^{21} & P'^{22} \end{pmatrix} + \lambda_2 \begin{pmatrix} Q'^{11} & Q'^{12} \\ Q'^{21} & Q'^{22} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Alternative:

$$\begin{pmatrix} a'^{11} & a'^{12} \\ a'^{21} & a'^{22} \end{pmatrix} = \mathbf{S}^{-1} \begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix} (\mathbf{S}^{-1})^T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

e) $\begin{pmatrix} b'^{11} & b'^{12} \\ b'^{21} & b'^{22} \end{pmatrix} = e^{i\theta\lambda_1} \begin{pmatrix} P'^{11} & P'^{12} \\ P'^{21} & P'^{22} \end{pmatrix} + e^{i\theta\lambda_2} \begin{pmatrix} Q'^{11} & Q'^{12} \\ Q'^{21} & Q'^{22} \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$

$$\begin{pmatrix} b^{11} & b^{12} \\ b^{21} & b^{22} \end{pmatrix} = \mathbf{S} \begin{pmatrix} b'^{11} & b'^{12} \\ b'^{21} & b'^{22} \end{pmatrix} \mathbf{S}^T = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$$

Alternative:

$$\begin{pmatrix} b^{11} & b^{12} \\ b^{21} & b^{22} \end{pmatrix} = e^{i\theta\lambda_1} \begin{pmatrix} P^{11} & P^{12} \\ P^{21} & P^{22} \end{pmatrix} + e^{i\theta\lambda_2} \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{i\theta} + e^{-i\theta} & e^{i\theta} - e^{-i\theta} \\ e^{i\theta} - e^{-i\theta} & e^{i\theta} + e^{-i\theta} \end{pmatrix}$$

3 Lokale Transformation [30 Punkte]

a) Transformation der Basis: $\mathbf{dx} = dx^i \mathbf{e}_i = dx'^j (\partial'_j x^i) \mathbf{e}_i = dx'^j \mathbf{e}'_j \rightarrow \mathbf{e}'_j = \partial'_j x^i \mathbf{e}_i \quad (\partial'_j x^i = \frac{\partial x^i}{\partial x'^j})$

$$\mathbf{S} = (s^i{}_j) = (\partial'_j x^i) = \begin{pmatrix} v\cos\phi & u\cos\phi & -uv\sin\phi \\ v\sin\phi & u\sin\phi & uv\cos\phi \\ u & -v & 0 \end{pmatrix}$$

b)

$$\mathbf{g}' = \mathbf{S}^T\mathbf{S} \text{ (oder } g'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j) \rightarrow \mathbf{g}' = \begin{pmatrix} u^2+v^2 & 0 & 0 \\ 0 & u^2+v^2 & 0 \\ 0 & 0 & (uv)^2 \end{pmatrix}$$

$$\text{und } \mathbf{g}'^* = \mathbf{g}'^{-1} = \begin{pmatrix} 1/(u^2+v^2) & 0 & 0 \\ 0 & 1/(u^2+v^2) & 0 \\ 0 & 0 & 1/(uv)^2 \end{pmatrix}$$

Alternative Lösung für $\mathbf{g}'^*$:

$$\begin{pmatrix} \mathbf{e}'^1 \\ \mathbf{e}'^2 \\ \mathbf{e}'^3 \end{pmatrix} = \mathbf{S}^{-1} \begin{pmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \end{pmatrix}$$

$$(g'^{ij}) = (\mathbf{e}'^i \cdot \mathbf{e}'^j) = \mathbf{S}^{-1} \begin{pmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \end{pmatrix} \begin{pmatrix} \mathbf{e}^{1T} & \mathbf{e}^{2T} & \mathbf{e}^{3T} \end{pmatrix} (\mathbf{S}^{-1})^T = \mathbf{S}^{-1}(\mathbf{S}^{-1})^T = (\mathbf{S}^T\mathbf{S})^{-1} = \mathbf{g}'^{-1}$$

$$= \begin{pmatrix} 1/(u^2+v^2) & 0 & 0 \\ 0 & 1/(u^2+v^2) & 0 \\ 0 & 0 & 1/(uv)^2 \end{pmatrix}$$

c) In der dualen Basis $\mathbf{b}^T = -x^2 y \mathbf{e}^1 + x^3 \mathbf{e}^2 = \begin{pmatrix} -x^2 y & x^3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \end{pmatrix} = \begin{pmatrix} -x^2 y & x^3 & 0 \end{pmatrix} \mathbf{S} \begin{pmatrix} \mathbf{e}'^1 \\ \mathbf{e}'^2 \\ \mathbf{e}'^3 \end{pmatrix}$

$$= \begin{pmatrix} -x^2 yv\cos\phi + x^3 v\sin\phi & -x^2 yu\cos\phi + x^3 u\sin\phi & x^2 yuv\sin\phi + x^3 uv\cos\phi \end{pmatrix} \begin{pmatrix} \mathbf{e}'^1 \\ \mathbf{e}'^2 \\ \mathbf{e}'^3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & u^4 v^4 \cos^2\phi \end{pmatrix} \begin{pmatrix} \mathbf{e}'^1 \\ \mathbf{e}'^2 \\ \mathbf{e}'^3 \end{pmatrix}$$

$\oint_C \mathbf{b} \cdot \mathbf{x} = \oint_C (u^4 v^4 \cos^2\phi \mathbf{e}'^3) \cdot (dx'^i \mathbf{e}'_i) = \oint_C u^4 v^4 \cos^2\phi \delta^3_i dx'^i = \oint_{C_1} u^4 v^4 \cos^2\phi dx'^3 = \int_0^{2\pi} \cos^2\phi d\phi = \frac{1}{2}\int_0^{2\pi} (\cos(2\phi)+1) d\phi = \pi$