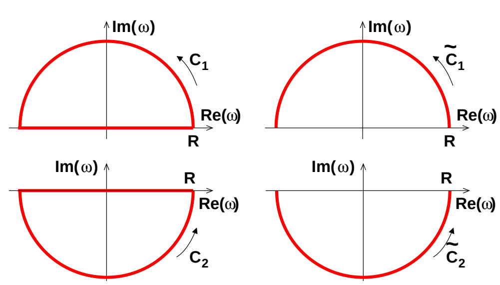
2. Test - Lösungen **18.1.2019**

1 Rechenbeispiele [30 Punkte, 6 Punkte je Frage]

a) $\int_0^\infty \delta(2x^2+x-1)\sin(\pi x)dx = \int_0^\infty \delta((2x-1)(x+1))\sin(\pi x)dx = \frac{1}{4x+1}\sin(\pi x)\Big|_{x=1/2} = \frac{1}{3}$
oder
$t = 2x^2+x-1 \to x = (\sqrt{8t+9}-1)/4 \ (x>0) \ dx = dt/\sqrt{8t+9}$
$\int_0^\infty \delta(2x^2+x-1)\sin(\pi x)dx = \int_{-9/8}^\infty \delta(t)\sin(\pi(\sqrt{8t+9}-1)/4)/\sqrt{8t+9}dt = \frac{1}{3}$
b) $\int_{-\infty}^\infty f'(x)\sin x dx = f(x)\sin x|_{-\infty}^\infty - \int_{-\infty}^\infty f(x)\cos x dx = -\int_{-\pi/2}^{\pi/2}\sin|x|\cos x dx$
$= -2\int_0^{\pi/2}\sin x\cos x dx = -1$
oder
$f(x) = H(\pi/2-x)H(x+\pi/2)\sin|x|$
$\to f'(x) = -\delta(\pi/2-x)H(x+\pi/2)\sin|x| + H(\pi/2-x)\delta(x+\pi/2)\sin|x| + H(\pi/2-x)H(x+\pi/2)\mathrm{sgn}(x)\cos x$
$= -\delta(\pi/2-x) + \delta(x+\pi/2) + H(\pi/2-x)H(x+\pi/2)\mathrm{sgn}(x)\cos x$
$\int_{-\infty}^\infty f'(x)\sin x dx = -\int_{-\infty}^\infty \delta(\pi/2-x)\sin x dx + \int_{-\infty}^\infty \delta(x+\pi/2)\sin x dx + \int_{-\infty}^\infty H(\pi/2-x)H(x+\pi/2)\mathrm{sgn}(x)\cos x\sin x dx$
$= -1-1+\int_{-\pi/2}^{\pi/2}\mathrm{sgn}(x)\cos x\sin x dx = -2+2\int_0^{\pi/2}\cos x\sin x dx = -1$
c) $t = x^2/2 \to x = \sqrt{2t} \ (x>0) \to dx = dt/\sqrt{2t}$
$\int_0^\infty x^4 e^{-x^2/2}dx = \int_0^\infty (2t)^2 e^{-t}/\sqrt{2t}dt = 2\sqrt{2}\int_0^\infty t^{3/2}e^{-t}dt = 2\sqrt{2}\Gamma(5/2) = 2\sqrt{2}(3/2)\Gamma(3/2)$
$= 2\sqrt{2}(3/2)(1/2)\Gamma(1/2) = (3/2)\sqrt{2\pi}$
d)$t = \sin^2\theta \to dt = 2\sin\theta\cos\theta d\theta = 2(1-t)^{1/2}t^{1/2}d\theta$
$\int_0^{\pi/2}\sin^3\theta d\theta = \int_0^1 t^{3/2}(1/2)(1-t)^{-1/2}t^{-1/2}dt = (1/2)\int_0^1 t(1-t)^{-1/2}dt = (1/2)B(2,1/2)$
$= (1/2)\Gamma(2)\Gamma(1/2)/\Gamma(5/2) = (1/2)/(3/4) = 2/3$
e)
$\partial_x \frac{x^2}{\Gamma(3)}H(x) = \partial_x \frac{x^2}{2}H(x) = xH(x) + \frac{x^2}{2}\delta(x) = xH(x)$
$\partial_x^2 \frac{x^2}{\Gamma(3)}H(x) = \partial_x xH(x) = H(x) + x\delta(x) = H(x)$
$\partial_x^3 \frac{x^2}{\Gamma(3)}H(x) = \partial_x H(x) = \delta(x)$

2 Greensche Funktion [30 Punkte]

a) Ansatz : $G_I(t,t') = (2\pi)^{-1}\int_{-\infty}^\infty \tilde{G}_I(\omega)\, e^{i\omega(t-t')}d\omega$
$\mathcal{L}_t G_I(t,t') = \delta(t-t') \to \frac{1}{2\pi}\int_{-\infty}^\infty(-\omega^2+2i\omega+2)\tilde{G}_I(\omega)\, e^{i\omega(t-t')}d\omega = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega(t-t')}d\omega$
Vergleich der Integranden: $(-\omega^2+2i\omega+2)\tilde{G}_I(\omega) = 1$
$\to \tilde{G}_I(\omega) = \frac{1}{-\omega^2+2i\omega+2} = -\frac{1}{(\omega+1-i)(\omega-1-i)}$
Fourier-Transformation $G_I(t,t') = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega = -\frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{i\omega(t-t')}}{(\omega+1-i)(\omega-1-i)}d\omega$
Integral mit dem Residuensatz
$\int_{-\infty}^\infty e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega = \lim_{R\to\infty} H(t-t')\left[\oint_{C_1} e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega - \int_{\tilde{C}_1} e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega\right]$
$-\lim_{R\to\infty} H(t'-t)\left[\oint_{C_2} e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega - \int_{\tilde{C}_2} e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega\right]$



Im Limes $R \to \infty$ $\int_{\tilde{C}_1} e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega \to 0$ und $\int_{\tilde{C}_2} e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega \to 0$
Beide Pole sind im oberen Halbkreis. Das Integral $\oint_{C_2}$ für $t < t'$ ist null.
Auf dem Halbkreis $\tilde{C}_1$,
$G_I(t,t') = \frac{1}{2\pi}\lim_{R\to\infty} H(t-t')\oint_{C_1} e^{i\omega(t-t')}\tilde{G}_I(\omega)d\omega = -\frac{1}{2\pi}\lim_{R\to\infty} H(t-t')\oint_{C_1} \frac{e^{i\omega(t-t')}}{(\omega+1-i)(\omega-1-i)}d\omega$
$= -iH(t-t')\left(-\frac{1}{2}e^{i(-1+i)(t-t')} + \frac{1}{2}e^{i(1+i)(t-t')}\right) = H(t-t')e^{-(t-t')}\sin(t-t')$
b) $x(t) = \int_{-\infty}^{\infty} G_I(t,t')H(t')dt' = \int_{-\infty}^{\infty} H(t-t')e^{-(t-t')}\sin(t-t')H(t')dt'$
Wenn $t < 0$, $x(t<0) = 0$
Wenn $t > 0$,
$x(t) = \int_0^t e^{-(t-t')}\sin(t-t')dt' = \frac{1}{2i}\int_0^t (e^{(i-1)(t-t')} - e^{(-i-1)(t-t')})dt' = \frac{1}{2i}\left(-\frac{1-e^{(i-1)t}}{i-1} - \frac{1-e^{(-i-1)t}}{i+1}\right)$
$= -\frac{1}{2}\left(\frac{1-e^{(i-1)t}}{2}(-1+i) + \frac{1-e^{(-i-1)t}}{2}(-1-i)\right) = -\frac{1}{2}\left(-1 - \frac{e^{(i-1)t}}{2}(-1+i) - \frac{e^{(-i-1)t}}{2}(-1-i)\right)$
$= \frac{1}{2} - \frac{1}{2}e^{-t}(\cos t + \sin t)$
$x(0) = 0$
$x'(t) = \frac{1}{2}e^{-t}(\cos t + \sin t) - \frac{1}{2}e^{-t}(-\sin t + \cos t) = e^{-t}\sin t \to x'(0) = 0.$

3 Differentialgleichung [40 Punkte]

a) $8uv\partial_v^2\Phi(u,v) + 3u^2v^{-1}\partial_u\Phi(u,v) + 6u\partial_v\Phi(u,v) + uv^{-1/2}\Phi(u,v) = 0$
Ansatz $\Phi(u,v) = P(u)Q(v)$:
$8uvP(u)Q''(v) + 3u^2v^{-1}P'(u)Q(v) + 6uP(u)Q'(v) + uv^{-1/2}P(u)Q(v) = 0$
$\times v/u$: $8v^2P(u)Q''(v) + 3uP'(u)Q(v) + 6vP(u)Q'(v) + v^{1/2}P(u)Q(v) = 0$
$\times 1/(PQ)$: $8v^2Q''(v)/Q(v) + 3uP'(u)/P(u) + 6vQ'(v)/Q(v) + v^{1/2} = 0$
$\to 3uP'(u)/P(u) = -8v^2Q''(v)/Q(v) - 6vQ'(v)/Q(v) - \sqrt{v} = Z$ (konstante)
$3uP'(u) - ZP(u) = 0$, $8v^2Q''(v) + 6vQ'(v) + (\sqrt{v} + Z)Q(v) = 0$
b) $v = x^2$, $Q(v) = y(x) \to \partial_v = \frac{dx}{dv}\partial_x = \frac{1}{2x}\partial_x$
$\partial_v Q(v) = \frac{1}{2x}\partial_x y(x)$, $\partial_v^2 Q(v) = \frac{1}{2x}\partial_x(\frac{1}{2x}\partial_x y(x)) = \frac{1}{2x}(-\frac{1}{2x^2}\partial_x y(x) + \frac{1}{2x}\partial_x^2 y(x)) = -\frac{1}{4x^3}\partial_x y(x) + \frac{1}{4x^2}\partial_x^2 y(x)$
$8v^2Q''(v) + 6vQ'(v) + (\sqrt{v} + Z)Q(v) = 0 \to 8x^4\left(-\frac{1}{4x^3}y'(x) + \frac{1}{4x^2}y''(x)\right) + 6x^2\frac{1}{2x}y'(x) + (x+Z)y(x) = 0 \to$
$2x^2y''(x) + xy'(x) + (x+Z)y(x) = 0$
c) Wenn $Z = 0$, $2xy'' + y' + y = 0$
Ansatz : $y = \sum_{n=0}^{\infty} a_n x^{n+\sigma}$
$\sum_{n=0}^{\infty} 2a_n(n+\sigma)(n+\sigma-1)x^{n+\sigma-1} + \sum_{n=0}^{\infty} a_n(n+\sigma)x^{n+\sigma-1} + \sum_{n=0}^{\infty} a_n x^{n+\sigma} = 0$
$\to \sum_{n=0}^{\infty} 2a_n(n+\sigma)(n+\sigma-1)x^{n+\sigma-1} + \sum_{n=0}^{\infty} a_n(n+\sigma)x^{n+\sigma-1} + \sum_{n=1}^{\infty} a_{n-1}x^{n+\sigma-1} = 0$
Koeffizientenvergleich, $x^{\sigma-1}$ Term : $2a_0\sigma(\sigma-1) + a_0\sigma = a_0\sigma(2\sigma-1) = 0$. Da $a_0 \neq 0$, $\sigma = 0, 1/2$
b) Koeffizientenvergleich
$x^{\sigma+n-1}$ Term $(n > 0)$: $2a_n(n+\sigma)(n+\sigma-1) + a_n(n+\sigma) + a_{n-1} = 0 \to a_n = -\frac{1}{(n+\sigma)(2n+2\sigma-1)}a_{n-1}$
Wenn $\sigma = 0$, $a_n = -\frac{1}{n(2n-1)}a_{n-1}$
$a_1 = -a_0$, $a_2 = -\frac{1}{2\cdot 3}a_1 = \frac{1}{2\cdot 3}a_0$, $a_3 = -\frac{1}{3\cdot 5}a_2 = -\frac{1}{(2\cdot 3)(3\cdot 5)}a_0, \cdots$
$a_n = (-1)^n\frac{1}{(2\cdot 3\cdots n)(3\cdot 5\cdots(2n-1))}a_0 = (-1)^n\frac{2^{n-1}(n-1)!}{n!(2n-1)!}a_0 = (-1)^n\frac{2^{n-1}}{n(2n-1)!}a_0 \left(\text{ oder } = (-1)^n\frac{2^n}{(2n)!}a_0\right)$
$y_1(x) = \sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(2n)!}a_0x^n = a_0\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}(\sqrt{2x})^{2n} = a_0\cos(\sqrt{2x})$
Wenn $\sigma = 1/2$, $a_n = -\frac{1}{n(2n+1)}a_{n-1}$
$a_1 = -\frac{1}{1\cdot 3}a_0$, $a_2 = -\frac{1}{2\cdot 5}a_1 = \frac{1}{(1\cdot 2)\cdot(3\cdot 5)}a_0$, $a_3 = -\frac{1}{3\cdot 7}a_2 = -\frac{1}{(1\cdot 2\cdot 3)(3\cdot 5\cdot 7)}a_0, \cdots$
$a_n = (-1)^n\frac{1}{(1\cdot 2\cdot 3\cdots n)(3\cdot 5\cdots(2n+1))}a_0 = (-1)^n\frac{2^n n!}{n!(2n+1)!}a_0 = (-1)^n\frac{2^n}{(2n+1)!}a_0$
$y_2(x) = \sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(2n+1)!}a_0x^{n+1/2} = a_0 2^{-1/2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}(\sqrt{2x})^{2n+1} = a_0 2^{-1/2}\sin(\sqrt{2x})$
$y(x) = A\cos(\sqrt{2x}) + B\sin(\sqrt{2x})$