2. Tutorium - Lösungen **19.10.2018**

- ANMERKUNG: Es liegt in der Verantwortung des Einzelnen, sich die Beispiele zunächst alleine und ganz ohne Hilfsmittel anzuschauen. Google, Wolfram Alpha, Lösungssammlungen, etc. helfen nur kurzfristig - leider nicht beim Test!

2.1 Kronecker-Delta

a) $\delta_{ii} = \delta_{11} + \delta_{22} + \cdots + \delta_{dd} = 1 + 1 + \cdots + 1 = d$, $\delta_{ii}\delta_{jj} = dd = d^2$, $\delta_{ij}\delta_{ji} = \delta_{ii} = d$
$\rightarrow \delta_{ii} + \delta_{ij}\delta_{ji} - \delta_{ii}\delta_{jj} = d + d - d^2 = 2d - d^2$
b) $x_i x_j \delta_{ji} = x_i x_i = 1 + 1 + 1 = 3$
c) $a_{ij}a_{ik}\delta_{jk} = a_{ij}a_{ij} = \mathrm{Tr}(\mathbf{A}\mathbf{A}^T) \rightarrow \mathbf{A}\mathbf{A}^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 4 & 0 \\ 3 & 0 & 9 \end{pmatrix}$
$\rightarrow \mathrm{Tr}(\mathbf{A}\mathbf{A}^T) = 15$

Alternative:

$a_{ij}a_{ik}\delta_{jk} = a_{ij}a_{ij} = \langle \mathbf{a}_j | \mathbf{a}_j \rangle$ mit $\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{pmatrix} \rightarrow \langle \mathbf{a}_j | \mathbf{a}_j \rangle = 1 + 4 + 10 = 15$

2.2 Transformationsmatrix

a) $\mathbf{T} = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$
b) $\mathbf{x} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 \end{pmatrix} \begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 \end{pmatrix} \mathbf{T} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$
Die Gleichung gilt für beliebigen Vektor (x^1, x^2). $\rightarrow \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 \end{pmatrix} \mathbf{T} \rightarrow \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \mathbf{T}^{-1}$
$\mathbf{S} = \mathbf{T}^{-1} (= \mathbf{T}^T) = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$
Alternative :
$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \mathbf{T}^{-1} \begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix}$
$\mathbf{x} = x^i \mathbf{e}_i = (\mathbf{T}^{-1})^i{}_j x'^j \mathbf{e}_i = x'^j \mathbf{e}'_j$ gilt für beliebigen Vektor $(x'^1, x'^2) \rightarrow \mathbf{e}'_j = (\mathbf{T}^{-1})^i{}_j \mathbf{e}_i \rightarrow \mathbf{S} = \mathbf{T}^{-1}$
c) $\begin{pmatrix} \langle \mathbf{e}'_1 | \mathbf{e}'_1 \rangle & \langle \mathbf{e}'_1 | \mathbf{e}'_2 \rangle \\ \langle \mathbf{e}'_2 | \mathbf{e}'_1 \rangle & \langle \mathbf{e}'_2 | \mathbf{e}'_2 \rangle \end{pmatrix} = \begin{pmatrix} \mathbf{e}'^T_1 \\ \mathbf{e}'^T_2 \end{pmatrix} \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 \end{pmatrix} = \mathbf{S}^T \underbrace{\begin{pmatrix} \mathbf{e}^T_1 \\ \mathbf{e}^T_2 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix}}_{=\mathbf{1}} \mathbf{S} = \underbrace{\mathbf{S}^T}_{=\mathbf{T}} \mathbf{S} = \mathbf{1}$

Alternative :
$\mathbf{e}'_i \cdot \mathbf{e}'_j = s^k{}_i \mathbf{e}_k \cdot s^\ell{}_j \mathbf{e}_\ell = s^k{}_i s^\ell{}_j \mathbf{e}_k \cdot \mathbf{e}_\ell = s^k{}_i s^\ell{}_j \delta_{k\ell} = s^k{}_i s^k{}_j = (\mathbf{S}^T \mathbf{S})_{ij} = \delta_{ij}$
d) $\begin{pmatrix} x''^1 \\ x''^2 \end{pmatrix} = \mathbf{T}_f \begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} \rightarrow \mathbf{T}_f = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ 1 & -\sqrt{3} \end{pmatrix} \rightarrow \mathbf{S}_f = \mathbf{T}_f^{-1} = \begin{pmatrix} 1 & 1 \\ 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$
Orthonormalität:
$\begin{pmatrix} \langle \mathbf{f}_1 | \mathbf{f}_1 \rangle & \langle \mathbf{f}_1 | \mathbf{f}_2 \rangle \\ \langle \mathbf{f}_2 | \mathbf{f}_1 \rangle & \langle \mathbf{f}_2 | \mathbf{f}_2 \rangle \end{pmatrix} = \mathbf{S}_f^T \underbrace{\begin{pmatrix} \mathbf{e}'^T_1 \\ \mathbf{e}'^T_2 \end{pmatrix} \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 \end{pmatrix}}_{=\mathbf{1} \quad \text{(Bsp.(c))}} \mathbf{S}_f = \mathbf{S}_f^T \mathbf{S}_f = \begin{pmatrix} 1 & 1/\sqrt{3} \\ 1 & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix}$
nicht orthogonal und nicht normiert
e) (x^1, x^2)-Koordinate:

$$\langle \mathbf{x}|\mathbf{x}\rangle = \begin{pmatrix} x^1 & x^2 \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix}}_{=\mathbf{1}} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} x^1 & x^2 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = x^i x^i$$

(oder $\langle \mathbf{x}|\mathbf{x}\rangle = x^i \mathbf{e}_i \cdot x^j \mathbf{e}_j = x^i x^j (\mathbf{e}_i \cdot \mathbf{e}_j) = x^i x^j \delta_{ij} = x^i x^i$)

(x'^1, x'^2)-Koordinate:

$$\langle \mathbf{x}|\mathbf{x}\rangle = \begin{pmatrix} x'^1 & x'^2 \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{e}_1'^T \\ \mathbf{e}_2'^T \end{pmatrix} \begin{pmatrix} \mathbf{e}_1' & \mathbf{e}_2' \end{pmatrix}}_{=\mathbf{1} \text{ (aus Bsp.(c))}} \begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} = x'^i x'^i$$

(oder $\langle \mathbf{x}|\mathbf{x}\rangle = x'^i \mathbf{e}_i' \cdot x'^j \mathbf{e}_j' = x'^i x'^j (\mathbf{e}_i' \cdot \mathbf{e}_j') = x'^i x'^j \delta_{ij} = x'^i x'^i$)

(x''^1, x''^2)-Koordinate:

$$\langle \mathbf{x}|\mathbf{x}\rangle = \begin{pmatrix} x''^1 & x''^2 \end{pmatrix} \begin{pmatrix} \mathbf{f}_1^T \\ \mathbf{f}_2^T \end{pmatrix} \begin{pmatrix} \mathbf{f}_1 & \mathbf{f}_2 \end{pmatrix} \begin{pmatrix} x''^1 \\ x''^2 \end{pmatrix} = \begin{pmatrix} x''^1 & x''^2 \end{pmatrix} \underbrace{\frac{1}{3}\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}}_{\text{siehe Bsp.(d)}} \begin{pmatrix} x''^1 \\ x''^2 \end{pmatrix}$$

$= \frac{4}{3}(x''^1 x''^1 + x''^1 x''^2 + x''^2 x''^2)$

(oder $\langle \mathbf{x}|\mathbf{x}\rangle = x''^i \mathbf{f}_i \cdot x''^j \mathbf{f}_j = x''^i x''^j \underbrace{(\mathbf{f}_i \cdot \mathbf{f}_j)}_{\text{Bsp.(d)}} = \frac{4}{3}(x''^1 x''^1 + x''^1 x''^2 + x''^2 x''^2)$)

2.3 Duale Basis

a) $\mathbf{e}_i'^T \cdot \mathbf{e}_j' = \delta_{ij} \rightarrow \mathbf{e}'^i = \mathbf{e}_i'^T$

b) Orthogonalität der dualen Basis $\begin{pmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \end{pmatrix} \begin{pmatrix} \mathbf{f}_1 & \mathbf{f}_2 \end{pmatrix} = \mathbf{1} \rightarrow \mathbf{S}_f^* \underbrace{\begin{pmatrix} \mathbf{e}'^1 \\ \mathbf{e}'^2 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1' & \mathbf{e}_2' \end{pmatrix}}_{=\mathbf{1}} \mathbf{S}_f = \mathbf{1} \rightarrow \mathbf{S}_f^* \mathbf{S}_f = \mathbf{1}$

$\mathbf{S}_f^* = \mathbf{S}_f^{-1} = \mathbf{T}_f = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ 1 & -\sqrt{3} \end{pmatrix} \rightarrow \mathbf{f}^1 = \frac{1}{2}(\mathbf{e}'^1 + \sqrt{3}\mathbf{e}'^2)$ und $\mathbf{f}^2 = \frac{1}{2}(\mathbf{e}'^1 - \sqrt{3}\mathbf{e}'^2)$

$\mathbf{x} = \begin{pmatrix} x_1'' & x_2'' \end{pmatrix} \begin{pmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \end{pmatrix} = \begin{pmatrix} x_1' & x_2' \end{pmatrix} \begin{pmatrix} \mathbf{e}'^1 \\ \mathbf{e}'^2 \end{pmatrix} \rightarrow \begin{pmatrix} x_1'' & x_2'' \end{pmatrix} \mathbf{S}_f^* \begin{pmatrix} \mathbf{e}'^1 \\ \mathbf{e}'^2 \end{pmatrix} = \begin{pmatrix} x_1' & x_2' \end{pmatrix} \begin{pmatrix} \mathbf{e}'^1 \\ \mathbf{e}'^2 \end{pmatrix}$

Da $\{\mathbf{e}'^1, \mathbf{e}'^2\}$ eine Orthonormalbasis ist, $\begin{pmatrix} x_1'' & x_2'' \end{pmatrix} \mathbf{S}_f^* = \begin{pmatrix} x_1' & x_2' \end{pmatrix}$

$\rightarrow \mathbf{T}_f^* = \mathbf{S}_f^{*-1} = \mathbf{S}_f = \mathbf{T}_f^{-1} = \begin{pmatrix} 1 & 1 \\ 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$

c) $x_i'' x''^i = x_j' t_f^{*j}{}_i x'^k t_f{}^i{}_k = x_j' x'^k t_f^{*j}{}_i t_f{}^i{}_k = x_j' x'^k (\mathbf{T}_f^* \mathbf{T}_f)^j{}_k = x_j' x'^k (\mathbf{T}_f^{-1} \mathbf{T}_f)^j{}_k = x_j' x'^k \delta^j{}_k = x_j' x'^j$