6. Tutorium - Lösungen **7.12.2018**

- ANMERKUNG: Es liegt in der Verantwortung des Einzelnen, sich die Beispiele zunächst alleine und ganz ohne Hilfsmittel anzuschauen. Google, Wolfram Alpha, Lösungssammlungen, etc. helfen nur kurzfristig - leider nicht beim Test!

6.1 Delta-Distribution und Heaviside-Funktion

a) $t = 2 - 3x \to x = (2-t)/3$ und $dx = (-1/3)dt$
$\int_{-\infty}^{\infty} \delta(2-3x)(3x^2+x-1)dx = \int_{\infty}^{-\infty} \delta(t)(3(2-t)^2/9 + (2-t)/3 - 1)(-1/3)dt$
$= \int_{-\infty}^{\infty} \delta(t)(1/3)((2-t)^2/3 + (2-t)/3 - 1)dt = (1/3)(4/3 + 2/3 - 1) = 1/3$

b) $t = 2x^2 + 3x - 2 \to x_1(t) = (-3 - \sqrt{25+8t})/4$ oder $x_2(t) = (-3 + \sqrt{25+8t})/4$
Für $-25/8 < t < \infty$, $-3/4 > x_1(t) > -\infty$ und $-3/4 < x_2(t) < \infty$
$\int_{-\infty}^{\infty} \delta(2x^2+3x-2)\log(x^2)dx = \int_{-\infty}^{-3/4} \delta(x^2-3x-4)\log(x^2)dx + \int_{-3/4}^{\infty} \delta(x^2-3x-4)\log(x^2)dx$
$= -\int_{-25/8}^{\infty} \delta(t)\log(x_1(t)^2)\frac{1}{4x_1(t)+3}dt + \int_{-25/8}^{\infty} \delta(t)\log(x_2(t)^2)\frac{1}{4x_2(t)+3}dt$
$= -\log(4)(-1/5) + \log(1/4)(1/5) = \log(1)/5 = 0$

c)
Kugelkoordinaten : $(x'^1, x'^2, x'^3) = (r, \theta, \phi)$ mit $(x^1, x^2, x^3) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ (siehe Bsp.5.1)
$\to dxdydz = dx'^1 dx'^2 dx'^3 |\det(\mathbf{S}')| = r^2 \sin\theta dr d\theta d\phi$ (Die Transformationsmatrix ist die Jacobi-Matrix.)
oder $dx'^1 dx'^2 dx'^3 |\det(\mathbf{S}')| = dx'^1 dx'^2 dx'^3 |\det(\mathbf{e}'_1\ \mathbf{e}'_2\ \mathbf{e}'_3)|$ ist das Volumen des von $dx'^1\mathbf{e}'_1$, $dx'^2\mathbf{e}'_2$, $dx'^3\mathbf{e}'_3$ gebildeten Parallelepipedes.
$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz H(R^2 - x^2 - y^2 - z^2) = \int_0^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi r^2 \sin\theta H(R^2 - r^2) = 4\pi \int_0^{\infty} dr r^2 H(R^2 - r^2)$
$= 4\pi \int_0^R dr r^2 = \frac{4}{3}\pi R^3$ (Volumen der Kugel mit Radius R)
Anmerkung:
$\frac{dI}{d(R^2)} = \int dxdydz\, \delta(R^2 - x^2 - y^2 - z^2) = \frac{dI}{d(R^2)}\frac{4}{3}\pi R^3 = \frac{dI}{dX}\frac{4}{3}\pi X^{3/2} = 2\pi X^{1/2} = 2\pi R$ (siehe Eq.(7.165) im Skriptum)

d) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz \delta\left(R - \sqrt{x^2+y^2+z^2}\right) = \int_0^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, r^2 \sin\theta\, \delta\left(R - r\right)$
$= 4\pi \int_0^{\infty} dr\, r^2 \delta\left(R - r\right) = 4\pi R^2$ (Oberfläche der Kugel mit Radius R)
Alternative :
$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz \delta\left(R - \sqrt{x^2+y^2+z^2}\right) = \frac{d}{dr}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz H\left(R - \sqrt{x^2+y^2+z^2}\right)$
$= \frac{d}{dR}\left(\frac{4}{3}\pi R^3\right) = 4\pi R^2$
Anmerkung : $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz\, H\left(R^2 - (x^2+y^2+z^2)\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz\, H\left(R - \sqrt{x^2+y^2+z^2}\right)$
aber $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz\, \delta\left(R^2 - (x^2+y^2+z^2)\right) \neq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz\, \delta\left(R - \sqrt{x^2+y^2+z^2}\right)$

6.2 Deltafolge

a) $\int_{-\infty}^{\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2}\varphi(x)dx = \int_{-\infty}^{\infty} \frac{n}{\sqrt{\pi}} e^{-y^2}\varphi\left(\frac{y}{n}\right)\frac{1}{n}dy = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2}\varphi\left(\frac{y}{n}\right)dy$
$\xrightarrow{n\to\infty} \varphi\left(0\right)\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2}dy = \varphi\left(0\right)$

b) $f_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \to x(d/dx)f_n(x) = -2\frac{n^3}{\sqrt{\pi}}x^2 e^{-n^2x^2}$
$\int_{-\infty}^{\infty} \left[x(d/dx)f_n(x)\right] e^{-n^2x^2}\varphi(x)dx = -2\int_{-\infty}^{\infty} \frac{n^3}{\sqrt{\pi}}x^2 e^{-n^2x^2}\varphi(x)dx = -2\int_{-\infty}^{\infty} \frac{n^3}{\sqrt{\pi}}\left(\frac{y}{n}\right)^2 e^{-y^2}\varphi\left(\frac{y}{n}\right)\frac{1}{n}dy$
$= -2\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} y^2 e^{-y^2}\varphi\left(\frac{y}{n}\right)dy \xrightarrow{n\to\infty} -2\varphi\left(0\right)\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} y^2 e^{-y^2}dy = -\varphi\left(0\right)$

c) $\int_{-\infty}^{\infty} f_n(x)e^{-ikx}dx = \int_{-\infty}^{\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2}e^{-ikx}dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2}e^{-i(k/n)y}dy = e^{-(k/n)^2/4} = e^{-k^2/(4n^2)} \xrightarrow{n\to\infty} 1$

6.3 Verallgemeinerte Funktion

a) $\left(\frac{d}{dt} + \gamma\right)\left(H(t)te^{-\gamma t}\right) = \delta(t)te^{-\gamma t} + H(t)e^{-\gamma t} = H(t)e^{-\gamma t}$
$\left(\frac{d}{dt} + \gamma\right)^2\left(H(t)te^{-\gamma t}\right) = \left(\frac{d}{dt} + \gamma\right)\left(H(t)e^{-\gamma t}\right) = \delta(t)e^{-\gamma t} - \gamma H(t)e^{-\gamma t} + \gamma H(t)e^{-\gamma t} = \delta(t)$

b) $\frac{\partial}{\partial t}H(t)\sqrt{\frac{1}{4\pi Dt}}e^{-x^2/(4Dt)} = \delta(t)\sqrt{\frac{1}{4\pi Dt}}e^{-x^2/(4Dt)} - \frac{1}{2t^{3/2}}H(t)\sqrt{\frac{1}{4\pi D}}e^{-x^2/(4Dt)} + \frac{x^2}{4Dt^2}H(t)\sqrt{\frac{1}{4\pi Dt}}e^{-x^2/(4Dt)}$

$= \delta(t)\delta(x) - \frac{1}{2t^{3/2}} H(t)\sqrt{\frac{1}{4\pi D}} e^{-x^2/(4Dt)} + \frac{x^2}{4Dt^2} H(t)\sqrt{\frac{1}{4\pi Dt}} e^{-x^2/(4Dt)}$
(siehe Bsp.6.2 : $\lim_{t\to 0} \sqrt{\frac{1}{4\pi Dt}} e^{-x^2/(4Dt)} = \delta(x)$)
$\frac{\partial^2}{\partial x^2} H(t)\sqrt{\frac{1}{4\pi Dt}} e^{-x^2/(4Dt)} = -\frac{\partial}{\partial x} H(t)\sqrt{\frac{1}{4\pi Dt}} \frac{x}{2Dt} e^{-x^2/(4Dt)} = -H(t)\sqrt{\frac{1}{4\pi Dt}} \frac{1}{2Dt} e^{-x^2/(4Dt)} + H(t)\sqrt{\frac{1}{4\pi Dt}} \frac{x^2}{4D^2t^2} e^{-x^2/(4Dt)}$
$\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) H(t)\sqrt{\frac{1}{4\pi Dt}} e^{-x^2/(4Dt)}$
$= \delta(t)\delta(x) - \frac{1}{2t^{3/2}} H(t)\sqrt{\frac{1}{4\pi D}} e^{-x^2/(4Dt)} + \frac{x^2}{4Dt^2} H(t)\sqrt{\frac{1}{4\pi Dt}} e^{-x^2/(4Dt)} + H(t)\sqrt{\frac{1}{4\pi Dt}} \frac{1}{2t} e^{-x^2/(4Dt)} - H(t)\sqrt{\frac{1}{4\pi Dt}} \frac{x^2}{4Dt^2} e^{-x^2/(4Dt)}$
$= \delta(t)\delta(x)$
c) $\frac{d}{dx}\sin|x| = \frac{d}{dx}\left(H(x)\sin x - H(-x)\sin x\right) = \delta(x)\sin x + H(x)\cos x + \delta(-x)\sin x - H(-x)\cos x$
$= H(x)\cos x - H(-x)\cos x = \mathrm{sgn}(x)\cos x$ (oder $\frac{d}{dx}\sin|x| = \frac{d|x|}{dx}\cos|x| = \mathrm{sgn}(x)\cos|x|$)
$\frac{d^2}{dx^2}\sin|x| = \frac{d}{dx}\left(H(x)\cos x - H(-x)\cos x\right) = \delta(x)\cos x - H(x)\sin x + \delta(-x)\cos x + H(-x)\sin x$
$= 2\delta(x) - \sin|x|$
(oder $\frac{d^2}{dx^2}\sin|x| = \frac{d}{dx}\mathrm{sgn}(x)\cos x = 2\delta(x)\cos x - \mathrm{sgn}(x)\sin x = 2\delta(x) - \mathrm{sgn}(x)\sin x$)
d) $\int_0^\infty f'(x)\sin x dx = f(x)\sin x|_{x=0}^\infty - \int_0^\infty f(x)\cos x dx = -\int_0^{\pi/2}\sin x\cos x dx = -\frac{1}{2}\int_0^{\pi/2}\sin 2x dx = -\frac{1}{2}$
Alternative Lösung
$f(x) = H(\pi/2+x)H(\pi/2-x)\sin x$
$\to f'(x) = H(\pi/2+x)H(\pi/2-x)\cos x + \delta(\pi/2+x)H(\pi/2-x)\sin x - H(\pi/2+x)\delta(\pi/2-x)\sin x$
$= H(\pi/2+x)H(\pi/2-x)\cos x - \delta(\pi/2+x)H(\pi/2-x) - H(\pi/2+x)\delta(\pi/2-x)$
$\int_0^\infty f'(x)\sin x dx$
$= \int_0^\infty H(\pi/2+x)H(\pi/2-x)\cos x\sin x dx - \int_0^\infty \delta(\pi/2+x)H(\pi/2-x)\sin x dx - \int_0^\infty H(\pi/2+x)\delta(\pi/2-x)\sin x dx$
$= \int_0^{\pi/2}\cos x\sin x dx - 0 - \int_0^\infty \delta(\pi/2-x)\sin x dx = -1/2$

6.4 Laplace-Operator

a) $\nabla^2\frac{1}{r} = \partial_i\partial_i\frac{1}{r} = -\partial_i\frac{x_i}{r^3} = -\frac{\delta_{ii}}{r^3} + \frac{3x_ix_i}{r^5} = -\frac{3}{r^3} + \frac{3r^2}{r^5} = 0$
b) Kugelkoordinaten : $(x'^1, x'^2, x'^3) = (r,\theta,\phi)$ mit $(x^1,x^2,x^3) = (r\sin\theta\cos\phi,\ r\sin\theta\sin\phi, r\cos\theta)$ (siehe Bsp.5.1)
$\nabla\frac{1}{r} = \mathbf{e}^i\partial_i\frac{1}{r} = \mathbf{e}'^i\partial'_i\frac{1}{r} = -\mathbf{e}'^1\frac{1}{r^2}$
Normaleneinheitsvektor $\mathbf{n} = \mathbf{e}'_1$ ($\mathbf{e}'_1\cdot\mathbf{e}'_1 = 1$ siehe Bsp.5.1)

$\int_{V_\varepsilon}\nabla\cdot\left[\left(\nabla\frac{1}{r}\right)\varphi(\mathbf{r})\right]d^3r \overset{\text{Gaußscher Integralsatz}}{=} \int_{S_\varepsilon}\left[\left(\nabla\frac{1}{r}\right)\varphi(\mathbf{r})\right]_{r=\varepsilon}\cdot\mathbf{n}\,dS = \int\left(-\mathbf{e}'^1\frac{1}{r^2}\varphi(\mathbf{r})\right)_{r=\varepsilon}\cdot\mathbf{e}'_1\,dS = -\int\frac{1}{\varepsilon^2}\ \varphi(\mathbf{r})|_{r=\varepsilon}\ dS$
(Anmerkung: $\nabla\frac{1}{r} = \mathbf{e}^i\partial_i(1/r) = \mathbf{e}'^i\partial'_i(1/r)$)
dS auf der Oberfläche S : $dS = \underbrace{|\mathbf{e}'_2\times\mathbf{e}'_3|d\theta d\phi}_{\text{Fläche der Parallelogramm}} = \varepsilon^2\sin\theta d\theta d\phi$

$\to \int_{V_\varepsilon}\nabla\cdot\left[\left(\nabla\frac{1}{r}\right)\varphi(\mathbf{r})\right]d^3r = -\int_0^{2\pi}\int_0^\pi \varphi(\mathbf{r})|_{r=\varepsilon}\sin\theta\,d\theta\,d\phi \xrightarrow[\varepsilon\to 0]{} -\varphi(0)\int_0^{2\pi}\int_0^\pi\sin\theta\,d\theta\,d\phi = -4\pi\varphi(0)$
c) V_c : das Volumen außerhalb der Kugel V_ε (d.h., $r > \varepsilon$)
$-\frac{1}{4\pi}\int d^3r\ \left(\nabla^2\frac{1}{r}\right)\varphi(\mathbf{r}) = -\frac{1}{4\pi}\int_{V_\varepsilon}d^3r\left(\nabla^2\frac{1}{r}\right)\varphi(\mathbf{r}) - \frac{1}{4\pi}\int_{V_c}d^3r\underbrace{\left(\nabla^2\frac{1}{r}\right)}_{=0}\varphi(\mathbf{r})$

$= -\frac{1}{4\pi}\int_{V_\varepsilon}d^3r\nabla\cdot\left[\left(\nabla\frac{1}{r}\right)\varphi(\mathbf{r})\right] + \frac{1}{4\pi}\int_{V_\varepsilon}d^3r\left(\nabla\frac{1}{r}\right)\left(\nabla\varphi(\mathbf{r})\right)$
Das zweite Integral verschwindet im Limes $\varepsilon\to 0$ für analytische Funktionen $\varphi(\mathbf{r})$,
$\int_{V_\varepsilon}d^3r\left(\nabla\frac{1}{r}\right)\left(\nabla\varphi(\mathbf{r})\right) = -\int_{V_\varepsilon}d^3r\,\mathbf{e}'^1\frac{1}{r^2}\mathbf{e}'_i\partial'^i\varphi(\mathbf{r}) = -\int_{V_\varepsilon}d^3r\,\frac{1}{r^2}\partial'^1\varphi(\mathbf{r})$
$= -\int_0^\varepsilon dr\int_0^\pi d\theta\int_0^{2\pi}d\phi\sin\theta\partial'^1\varphi(\mathbf{r}) \xrightarrow[\varepsilon\to 0]{} 0$
$(dV = d^3r = dx'^1dx'^2dx'^3|\det(\mathbf{e}'_1\ \mathbf{e}'_2\ \mathbf{e}'_3)| = r^2\sin\theta drd\theta d\phi)$

Aus dem Ergebnis von Bsp.b

$$-\frac{1}{4\pi}\int d^3r\ \left(\nabla^2\frac{1}{r}\right)\varphi(\mathbf{r}) \quad = \quad -\frac{1}{4\pi}\int_{V_\varepsilon}d^3r\nabla\cdot\left[\left(\nabla\frac{1}{r}\right)\varphi(\mathbf{r})\right] = \varphi(0)\,.$$

Deshalb ist $-\nabla^2 1/(4\pi r)$ eine dreidimensionale Delta-Distribution.
($-1/(4\pi r)$ ist die Greensche Funktion des Laplace-Operators.)