7. Tutorium - Lösungen **14.12.2018**

- ANMERKUNG: Es liegt in der Verantwortung des Einzelnen, sich die Beispiele zunächst alleine und ganz ohne Hilfsmittel anzuschauen. Google, Wolfram Alpha, Lösungssammlungen, etc. helfen nur kurzfristig - leider nicht beim Test!

7.1 Residuensatz

a) $\oint_C \frac{z}{2z^2-2z-4}dz = \oint_C \frac{z}{2(z+1)(z-2)}dz = 2\pi i \left(\left.\frac{z}{2(z+1)}\right|_{z=2} + \left.\frac{z}{2(z-2)}\right|_{z=-1} \right) = 2\pi i \left(\frac{1}{3}+\frac{1}{6}\right) = \pi i$

b) $\oint_{C_1} \frac{1}{4z^2+1}dz = \oint_{C_1} \frac{1}{4(z+i/2)(z-i/2)}dz = 2\pi i \left.\frac{1}{4(z+i/2)}\right|_{z=i/2} = 2\pi i \left(\frac{1}{4i}\right) = \frac{\pi}{2}$

c) $\oint_{C_2} \frac{1}{4z^2+1}dz = \oint_{C_2} \frac{1}{4(z+i/2)(z-i/2)}dz = 2\pi i \left.\frac{1}{4(z-i/2)}\right|_{z=-i/2} = 2\pi i \left(-\frac{1}{4i}\right) = -\frac{\pi}{2}$

d) $\oint_C \frac{2z^3+7z^2+8z+5}{2(z+1)^3}dz = \oint_C \frac{1}{2(z+1)^3}\left[2+(z+1)^2+2(z+1)^3\right]dz$

$= \oint_C \left[\frac{1}{(z+1)^3}+\frac{1}{2(z+1)}+1\right] = \pi i$

oder $\oint_C \frac{2z^3+7z^2+8z+5}{2(z+1)^3}dz = 2\pi i \frac{1}{2!}\frac{d^2}{dz^2}\frac{1}{2}(2z^3+7z^2+8z+5)\Big|_{z=-1} = 2\pi i \frac{1}{2!}\frac{1}{2}(12z+14)\Big|_{z=1} = \pi i$

e) $\oint_C \frac{e^{itz}}{z^2+4}dz = \oint_C \frac{e^{itz}}{(z+2i)(z-2i)}dz = 2\pi i \left.\frac{e^{itz}}{z+2i}\right|_{z=2i} = \frac{1}{2}\pi e^{-2t}$ (wenn $R>2$)

$\left|\int_{C_1}\frac{e^{itz}}{z^2+4}dz\right| = \left|\int_0^\pi \frac{e^{itR\exp i\theta}}{R^2e^{2i\theta}+4}iRe^{i\theta}d\theta\right| \le \int_0^\pi \left|\frac{e^{itR\exp i\theta}}{R^2e^{2i\theta}+4}iRe^{i\theta}\right|d\theta = \int_0^\pi \frac{R|e^{itR\exp i\theta}|}{|R^2e^{2i\theta}+4|}d\theta$

Da $|e^{itR\exp i\theta}| = |e^{itR\cos\theta}||e^{-tR\sin\theta}| = |e^{-tR\sin\theta}| \overset{R\to\infty}{\longrightarrow} 0$ wenn $t>0$ und $\sin\theta>0$ (d.h. $0<\theta<\pi$)

und $\frac{R}{|R^2e^{2i\theta}+4|} \overset{R\to\infty}{\longrightarrow} 0$, $\int_0^\pi \frac{R|e^{itR\exp i\theta}|}{|R^2e^{2i\theta}+4|}d\theta \overset{R\to\infty}{\longrightarrow} 0$

$\int_{-\infty}^{\infty}\frac{e^{itx}}{x^2+4}dx = \lim_{R\to\infty}\left(\oint_C \frac{e^{itz}}{z^2+4}dz - \int_{C_1}\frac{e^{itz}}{z^2+4}dz\right) = \frac{1}{2}\pi e^{-2t}$

7.2 Greensche Funktion I

a) Homogene Differentialgleichung: $\mathcal{L}_t x_0(t) = \frac{d^2}{dt^2}x_0(t) = 0$

Allgemeine Lösung : $x_0(t) = At + B$

b) Die Greensche Funktion $G(t,t')$ erfüllt die inhomogene Differentialgleichung $\mathcal{L}_t G(t,t') = \delta(t-t')$. Weil die Greensche Funktion die homogene Gleichung $\mathcal{L}_t G(t,t') = 0$ außer $t=t'$ erfüllt, nehmen wir den Ansatz

$G(t,t') = \begin{cases} A_1(t')t + B_1(t') & (t<t') \\ A_2(t')t + B_2(t') & (t'<t) \end{cases}$

iii) Translationsinvarianz (wenn a_i in $\mathcal{L}_t = \sum_i a_i d^i/dx^i$ konstante sind)

$G(t,t') = G(t-t',0) = \begin{cases} A_1(0)(t-t') + B_1(0) & (t<t') \\ A_2(0)(t-t') + B_2(0) & (t'<t) \end{cases}$

iv) $A_1(0)=\alpha$, $B_1(0)=\beta$

$G(t,t') = \begin{cases} \alpha(t-t') + \beta & (t<t') \\ A_2(0)(t-t') + B_2(0) & (t'<t) \end{cases}$

i) Stetigkeit der Greenschen Funktion

$\lim_{t\to t'^-} G(t,t') = \lim_{t\to t'^+} G(t,t') \to B_2(0) = \beta$

ii) $\frac{d}{dt}G(t,t')\Big|_{t=t'-\varepsilon}^{t=t'+\varepsilon} = 1$ $(\varepsilon>0) \to A_2(t'+\varepsilon) - A_1(t'-\varepsilon) = 1 \overset{\varepsilon\to 0}{\longrightarrow} A_2(0) = \alpha+1$

$\to G(t,t') = \begin{cases} \alpha(t-t') + \beta & (t<t') \\ \alpha(t-t') + \beta + (t-t') & (t'<t) \end{cases}$

Anmerkung : $G(t,t') = \alpha(t-t') + \beta$ ist die allgemeine Lösung der homogenen Differentialgleichung und $G(t.t') = H(t-t')(t-t')$ eine Lösung der inhomogenen Differentialgleichung.

c) $x(t) = \int_{-\infty}^{\infty} G(t,t')H(t')H(T-t')dt' = \int_0^T G(t,t')dt'$

Wenn $t \le 0$, $x(t) = \int_0^T (\alpha(t-t')+\beta)\,dt' = -\frac{\alpha}{2}T^2 + \alpha t T + \beta T$

$\lim_{t\to 0^-} x(t) = -\frac{\alpha}{2}T^2 + \beta T = 0 \to \beta = \frac{\alpha}{2}T$

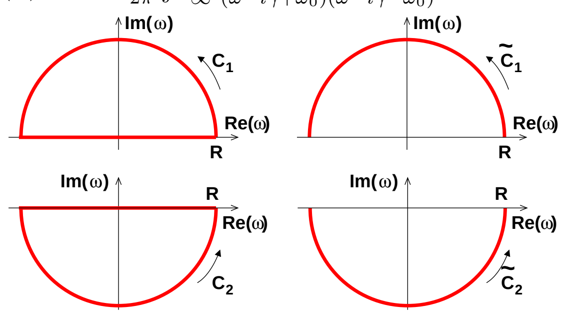
Wenn $0 < t \le T$, $x(t) = \int_0^t (\alpha(t-t') + \beta + (t-t'))\,dt' + \int_t^T (\alpha(t-t') + \beta)\,dt' = \int_0^t (t-t')dt' + \int_0^T (\alpha(t-t') + \beta)\,dt' = \frac{1}{2}t^2 - \frac{\alpha}{2}T^2 + \alpha t T + \beta T = \frac{1}{2}t^2 + \alpha t T$
$\lim_{t \to 0^+} x(t) = 0$
$\lim_{t \to T^-} x(t) = \frac{1}{2}T^2 + \alpha T^2 = 0 \to \alpha = -\frac{1}{2}$
Wenn $t > T$, $x(t) = \int_0^T (\alpha(t-t') + \beta + (t-t'))\,dt' = -\frac{\alpha}{2}T^2 + \alpha t T + \beta T + \frac{1}{2}T^2 = \alpha t T + \frac{1}{2}T^2 = \frac{1}{2}T(T-t)$
$\lim_{t \to T^+} x(t) = 0$

$$\to x(t) = \begin{cases} (1/2)tT & (t \le 0) \\ (1/2)t(t-T) & (0 < t \le T) \\ (1/2)T(t-T) & (T < t) \end{cases}$$

7.3 Greensche Funktion II

a) Ansatz : $G_I(t,t') = (2\pi)^{-1} \int_{-\infty}^{\infty} \tilde{G}_I(\omega)\, e^{i\omega(t-t')} d\omega$
$\mathcal{L}_t G_I(t,t') = \delta(t-t') \to \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega + \gamma + i\omega_0)(i\omega + \gamma - i\omega_0)\tilde{G}_I(\omega)\, e^{i\omega(t-t')} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t')} d\omega$
Vergleich der Integranden: $(i\omega + \gamma + i\omega_0)(i\omega + \gamma - i\omega_0)\tilde{G}_I(\omega) = 1$
$\to \tilde{G}_I(\omega) = \frac{1}{(i\omega+\gamma+i\omega_0)(i\omega+\gamma-i\omega_0)} = -\frac{1}{(\omega-i\gamma+\omega_0)(\omega-i\gamma-\omega_0)}$
b) Fourier-Transformation $G_I(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega(t-t')}}{(\omega-i\gamma+\omega_0)(\omega-i\gamma-\omega_0)} d\omega$

Das Integral kann mit $\int_{-\infty}^{\infty} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega = \lim_{R\to\infty}\left[\oint_{C_1} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega - \int_{\tilde{C}_1} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega\right]$ oder $\lim_{R\to\infty}\left[-\oint_{C_2} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega + \int_{\tilde{C}_2} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega\right]$ gerechnet werden. $\tilde{C}_1$ ist der obere Halbkreis und $\tilde{C}_2$ der untere Halbkreis. C_1 und C_2 sind die geschlossenen Halbkreise (siehe Abb.).



Auf dem Halbkreis $\tilde{C}_1$
$\left|\int_{\tilde{C}_1} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega\right| = \frac{1}{2\pi}\left|\int_0^\pi \frac{e^{iRe^{i\theta}(t-t')}}{(Re^{i\theta}-i\gamma+\omega_0)(Re^{i\theta}-i\gamma-\omega_0)} iRe^{i\theta} d\theta\right| < \frac{1}{2\pi}\int_0^\pi \left|\frac{e^{iRe^{i\theta}(t-t')}}{(Re^{i\theta}-i\gamma+\omega_0)(Re^{i\theta}-i\gamma-\omega_0)} iRe^{i\theta}\right| d\theta$
$= \frac{1}{2\pi}\int_0^\pi \left|\frac{R}{(Re^{i\theta}-i\gamma+\omega_0)(Re^{i\theta}-i\gamma-\omega_0)} R\right| |e^{-\mathrm{Im}(Re^{i\theta})(t-t')}| d\theta \to 0$ wenn $R \to \infty$ und $t > t'$.
In ähnlicher Weise $\left|\int_{\tilde{C}_2} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega\right| \to 0$ wenn $R \to \infty$ und $t < t'$.
Endlich $\int_{-\infty}^{\infty} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega = H(t-t')\lim_{R\to\infty}\oint_{C_1} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega - H(t'-t)\lim_{R\to\infty}\oint_{C_2} e^{i\omega(t-t')}\tilde{G}_I(\omega) d\omega$
Mit Hilfe des Residuensatzes und im Limes $R \to \infty$,
$G_I(t,t') = -iH(t-t')\left.\frac{e^{i\omega(t-t')}}{\omega-i\gamma+\omega_0}\right|_{\omega=\omega_0+i\gamma} - iH(t-t')\left.\frac{e^{i\omega(t-t')}}{\omega-i\gamma-\omega_0}\right|_{\omega=-\omega_0+i\gamma}$
$= -iH(t-t')\left(\frac{e^{i\omega_0(t-t')-\gamma(t-t')}}{2\omega_0} - \frac{e^{-i\omega_0(t-t')-\gamma(t-t')}}{2\omega_0}\right) = H(t-t')\frac{1}{\omega_0}e^{-\gamma(t-t')}\sin(\omega_0(t-t'))$
c) $x(t) = \int_{-\infty}^{\infty} G_I(t,t')f(t')dt' = \int_{-\infty}^{\infty} H(t-t')\frac{1}{\omega_0}e^{-\gamma(t-t')}\sin(\omega_0(t-t'))H(t')dt'$
$x(t<0) = 0$
$x(t>0) = \int_0^t \frac{1}{\omega_0}e^{-\gamma(t-t')}\sin(\omega_0(t-t'))dt' = \frac{1}{2i\omega_0}\int_0^t (e^{(-\gamma+i\omega_0)(t-t')} - e^{(-\gamma-i\omega_0)(t-t')})dt'$
$= \frac{1}{2i\omega_0}\left(-\frac{e^{(-\gamma+i\omega_0)(t-t')}}{-\gamma+i\omega_0} + \frac{e^{(-\gamma-i\omega_0)(t-t')}}{-\gamma-i\omega_0}\right)_0^t = \frac{1}{2i\omega_0}\left(-\frac{1}{-\gamma+i\omega_0} + \frac{1}{-\gamma-i\omega_0}\right) - \frac{1}{2i\omega_0}\left(-\frac{e^{(-\gamma+i\omega_0)t}}{-\gamma+i\omega_0} + \frac{e^{(-\gamma-i\omega_0)t}}{-\gamma-i\omega_0}\right)$
$= \frac{1}{\gamma^2+\omega_0^2} - \frac{e^{-\gamma t}}{\gamma^2+\omega_0}\left(\cos(\omega_0 t) + \frac{\gamma}{\omega_0}\sin(\omega_0 t)\right)$
$\to x(0) = 0$ und $\lim_{t\to\infty} x(t) = \frac{1}{\gamma^2+\omega_0^2}$
Anmerkung: alternative Lösung wie im Bsp.7.2
Allgemeine Lösung der homogenen Gleichung :
$x_0(t) = Ae^{-\gamma t}e^{i\omega_0 t} + Be^{-\gamma t}e^{-i\omega_0 t}$
Ansatz der Greenschen Funktion

$$G(t,t') = \begin{cases} A_1(t')e^{-\gamma t}e^{i\omega_0 t} + B_1(t')e^{-\gamma t}e^{-i\omega_0 t} & (t < t') \\ A_2(t')e^{-\gamma t}e^{i\omega_0 t} + B_2(t')e^{-\gamma t}e^{-i\omega_0 t} & (t > t') \end{cases}$$

Translationsinvarianz :

$$G(t,t') = G(t-t',0) = \begin{cases} A_1(0)e^{-\gamma(t-t')}e^{i\omega_0(t-t')} + B_1(0)e^{-\gamma(t-t')}e^{-i\omega_0(t-t')} & (t < t') \\ A_2(0)e^{-\gamma(t-t')}e^{i\omega_0(t-t')} + B_2(0)e^{-\gamma(t-t')}e^{-i\omega_0(t-t')} & (t > t') \end{cases}$$

Stetigkeit der Greenschen Funktion : $(A_1(0) - A_2(0)) + (B_1(0) - B_2(0)) = 0$

$\frac{d}{dt}G(t,t')\Big|_{t=t'-\varepsilon}^{t=t'+\varepsilon} = 1 \ (\varepsilon > 0)$

$\rightarrow A_2(0)(i\omega_0 - \gamma) + B_2(0)(-i\omega_0 - \gamma) - A_1(0)(i\omega_0 - \gamma) - B_1(0)(-i\omega_0 - \gamma) = 1$

$\rightarrow A_2(0) = A_1(0) + 1/(2i\omega_0), \ B_2(t') = B_1(t') - 1/(2i\omega_0)$

$$G(t,t') = \begin{cases} A_1(0)e^{-\gamma(t-t')}e^{i\omega_0(t-t')} + B_1(0)e^{-\gamma(t-t')}e^{-i\omega_0(t-t')} & (t < t') \\ A_1(0)e^{-\gamma(t-t')}e^{i\omega_0(t-t')} + B_1(0)e^{-\gamma(t-t')}e^{-i\omega_0(t-t')} + \frac{1}{\omega_0}e^{-\gamma(t-t')}\sin(\omega_0(t-t')) & (t > t') \end{cases}$$

$x(t<0) = \int_0^\infty G(t,t')dt' = \lim_{T\to\infty}\int_0^T G(t,t')dt'$

$= \lim_{T\to\infty} A_1(0)e^{-\gamma t}e^{i\omega_0 t}\frac{e^{\gamma T}e^{-i\omega_0 T}-1}{\gamma - i\omega_0} + \lim_{T\to\infty} B_1(0)e^{-\gamma t}e^{-i\omega_0 t}\frac{e^{\gamma T}e^{i\omega_0 T}-1}{\gamma + i\omega_0}$

$\lim_{t\to 0^-} x(t) = \lim_{T\to\infty} A_1(0)\frac{e^{\gamma T}e^{-i\omega_0 T}-1}{\gamma - i\omega_0} + \lim_{T\to\infty} B_1(0)\frac{e^{\gamma T}e^{i\omega_0 T}-1}{\gamma + i\omega_0} = 0 \rightarrow A_1(0) = A_2(0) = 0 \ x(t<0) = 0$

$x(t>0) = \int_0^t \frac{1}{\omega_0}e^{-\gamma(t-t')}\sin(\omega_0(t-t'))dt' = \frac{1}{\gamma^2+\omega_0^2} - \frac{e^{-\gamma t}}{\gamma^2+\omega_0}\left(\cos(\omega_0 t) + \frac{\gamma}{\omega_0}\sin(\omega_0 t)\right)$