1. Test - Lösungen **29.11.2019**

1 Rechenbeispiele [30 Punkte, 6 Punkte je Frage]

a) $\partial_i x_i/(x_j x_j) = \delta_{ii}/(x_j x_j) + x_i \partial_i(1/(x_j x_j)) = 3/(x_j x_j) + x_i \partial_i(1/(x_j x_j)) = 3/(x_j x_j) - x_i/(x_j x_j)^2 \partial_i(x_k x_k)$
$= 3/(x_j x_j) - x_i/(x_j x_j)^2 2x_k \delta_{ik} = 3/(x_j x_j) - 2x_i x_i/(x_j x_j)^2 = 1/(x_j x_j)$
b) $\mathbf{f}^1 \cdot \mathbf{f}_2 = 0 \rightarrow \mathbf{f}^1 = C_1(1\,2)$, $\quad \mathbf{f}^1 \cdot \mathbf{f}_1 = 5C_1 = 1 \rightarrow C_1 = 1/5 \rightarrow \mathbf{f}^1 = (1/5\;2/5)$
$\mathbf{f}^2 \cdot \mathbf{f}_1 = 0 \rightarrow \mathbf{f}^2 = C_2(2\;-1)$, $\quad \mathbf{f}^2 \cdot \mathbf{f}_2 = -5C_2 = 1 \rightarrow \mathbf{e}^2 = (-2/5\;1/5)$
alternative Lösung:
$$\begin{pmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \end{pmatrix} \begin{pmatrix} \mathbf{f}_1 & \mathbf{f}_2 \end{pmatrix} = \mathbf{I} \rightarrow \begin{pmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 & \mathbf{f}_2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix}^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \end{pmatrix}$$
c) $\det(\mathbf{A}) = \varepsilon_{n\ell m} a_{n1} a_{\ell 2} a_{m3} = \varepsilon_{n\ell m} (\mathbf{e}_i)_n (\mathbf{e}_j)_\ell (\mathbf{e}_k)_m = \varepsilon_{n\ell m} \delta_{in} \delta_{j\ell} \delta_{km} = \varepsilon_{ijk}$
Alternative Lösung:

Wenn $i = j$ oder $j = k$ oder $k = i$, z.B. $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \det(\mathbf{A}) = 0$

Wenn i, j, k eine gerade Permutation von 1,2,3 ist, z.B. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \det(\mathbf{A}) = 1$

Wenn i, j, k eine ungerade Permutation von 1,2,3 ist, z.B. $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \det(\mathbf{A}) = -1$

$\rightarrow \det(\mathbf{A}) = \varepsilon_{ijk}$.
d) Da $\mathbf{g}^* = \mathbf{g}^{-1}$ und $g_{ij} = g_{ji}$, $g^{ij} g_{ij} = (\mathbf{g}^* \mathbf{g}^T)_{ii} = (\mathbf{g}^* \mathbf{g})_{ii} = \mathrm{Tr}(\mathbf{1}) = d$
e) $\mathbf{a} \cdot \mathbf{b} = 0 \rightarrow$ orthogonal $\rightarrow \mathbf{E}_b \mathbf{E}_a = 0 \rightarrow \mathbf{E}_b \mathbf{E}_a^2 \mathbf{x} = 0$
Alternative Lösung:
$\mathbf{E}_a = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ und $\mathbf{E}_b = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$
$\mathbf{E}_a \mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$
$\mathbf{E}_a^2 \mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$
$\mathbf{E}_b \mathbf{E}_a^2 \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2 Tensoren [40 Punkte]

a) Eigenwertgleichung: $\begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} -\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$

$\rightarrow \det \begin{pmatrix} -\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{pmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0 \rightarrow \lambda_1 = -1$ und $\lambda_2 = 2$

b) Eigenvektor: $\mathbf{e}'_i = s^j{}_i \mathbf{e}_j = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \begin{pmatrix} s^1{}_i \\ s^2{}_i \end{pmatrix}$

$\begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix} \begin{pmatrix} s^1{}_i \\ s^2{}_i \end{pmatrix} = \begin{pmatrix} \sqrt{2} s^2{}_i \\ \sqrt{2} s^1{}_i + s^2{}_i \end{pmatrix} = \lambda \begin{pmatrix} s^1{}_i \\ s^2{}_i \end{pmatrix} \rightarrow \sqrt{2} s^2{}_i = \lambda s^1{}_i$ und $\sqrt{2} s^1{}_i + s^2{}_i = \lambda s^2{}_i$

Wenn $\lambda_1 = -1$, $\begin{pmatrix} s^1{}_1 \\ s^2{}_1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} \left(\text{oder } \frac{1}{\sqrt{3}} \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} \right)$. Wenn $\lambda_2 = 2$, $\begin{pmatrix} s^1{}_2 \\ s^2{}_2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$.

oder $= -\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$.

c) $\mathbf{T} = \mathbf{S}^{-1} = \mathbf{S}^T \rightarrow \mathbf{e}_i = t^j{}_i \mathbf{e}'_j$
$\mathbf{A} = a^{ij} |\mathbf{e}_i\rangle\langle e_j| = a^{ij} t^k{}_i |\mathbf{e}'_k\rangle\langle e'_\ell| t^\ell{}_j = t^\ell{}_j a^{ij} t^k{}_i |\mathbf{e}'_k\rangle\langle e'_\ell| \equiv a'^{ij} |\mathbf{e}'_k\rangle\langle e'_\ell|$

$\rightarrow (a'^{ij}) = \mathbf{T}(a^{ij})\mathbf{T}^T = \mathbf{S}^T(a^{ij})\mathbf{S} = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}\frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

d) $\mathbf{B} = \mathbf{A}^n$ in der Eigenbasis : $(b'^{ij}) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}^n = (b'^{ij}) = \begin{pmatrix} (-1)^n & 0 \\ 0 & 2^n \end{pmatrix}$

$\mathbf{B}$ in der kartesichen Basis : $(b^{ij}) = \mathbf{S}\begin{pmatrix} (-1)^n & 0 \\ 0 & 2^n \end{pmatrix}\mathbf{S}^T = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 2^n \end{pmatrix}\frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix} =$
$\frac{1}{3}\begin{pmatrix} 2(-1)^n + 2^n & \sqrt{2}((-1)^{n+1} + 2^n) \\ \sqrt{2}((-1)^{n+1} + 2^n) & (-1)^n + 2^{n+1} \end{pmatrix}$

Alternative Lösung 1: $(a^{ij})^2 = (a^{ij}a^{jk}) = (s^i{}_\ell a'^{\ell m} \underbrace{s^j{}_m s^j{}_n}_{\mathbf{S}^T\mathbf{S}=1} a'^{no} s^k{}_o) = (s^i{}_\ell a'^{\ell m}\delta_{mn}a'^{no}s^k{}_o) = (s^i{}_\ell a'^{\ell m}a'^{mo}s^k{}_o)$

$\rightarrow (a^{ij})^n = (s^i{}_\ell a'^{\ell m}a'^{mo}\cdots a'^{pq}s^k{}_q) = S(a'^{ij})^n S^T = \frac{1}{3}\begin{pmatrix} 2(-1)^n + 2^n & \sqrt{2}((-1)^{n+1} + 2^n) \\ \sqrt{2}((-1)^{n+1} + 2^n) & (-1)^n + 2^{n+1} \end{pmatrix}$

Alternative Lösung 2:
Projektor $\mathbf{E}_i = |\mathbf{e}'_i\rangle\langle\mathbf{e}'_i|$

In der kartesichen Basis $\mathbf{E}_1 = \frac{1}{3}\begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$ und $\mathbf{E}_2 = \frac{1}{3}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$

$\mathbf{A} = \lambda_i\mathbf{E}_i$
$\mathbf{A}^2 = \lambda_i\mathbf{E}_i\lambda_j\mathbf{E}_j = \lambda_i\lambda_j\delta_{ij}\mathbf{E}_i = (\lambda_i)^2\mathbf{E}_i$
$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = (\lambda_i)^2\mathbf{E}_i\lambda_j\mathbf{E}_j = (\lambda_i)^2\lambda_j\delta_{ij}\mathbf{E}_i = (\lambda_i)^3\mathbf{E}_i$
$\mathbf{A}^n = (\lambda_i)^n\mathbf{E}_i = (-1)^n\frac{1}{3}\begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} + 2^n\frac{1}{3}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$
$(\mathbf{A}^n = \mathbf{A}^{n-1}\mathbf{A} = (\lambda_i)^{n-1}\mathbf{E}_i\lambda_j\mathbf{E}_j = (\lambda_i)^n\mathbf{E}_i)$

d) $\mathbf{x}(t) = \exp(\mathbf{A}t)\,\mathbf{x}(0) = \sum_{n=0}^{\infty}\frac{\mathbf{A}^n}{n!}\mathbf{x}(0) = \sum_{n=0}^{\infty}\sum_{i=1}^{2}\frac{\lambda_i^n}{n!}\mathbf{E}_i\mathbf{x}(0) = \sum_{i=1}^{2}e^{\lambda_i t}\mathbf{E}_i\mathbf{x}(0) = \sum_{i=1}^{2}e^{\lambda_i t}\mathbf{E}_i\mathbf{e}_1 =$
$e^{-t}\frac{1}{3}(2\mathbf{e}_1 - \sqrt{2}\mathbf{e}_2) + e^{2t}\frac{1}{3}(\mathbf{e}_1 + \sqrt{2}\mathbf{e}_2) = \frac{1}{3}(2e^{-t} + e^{2t})\mathbf{e}_1 + \frac{\sqrt{2}}{3}(-e^{-t} + e^{2t})\mathbf{e}_2$

3 Lokale Transformation [30 Punkte]

a) Transformation der Basis: $\mathbf{dx} = dx^i\mathbf{e}_i = dx'^j(\partial'_j x^i)\mathbf{e}_i = dx'^j\mathbf{e}'_j \rightarrow \mathbf{e}'_j = \partial'_j x^i\mathbf{e}_i \quad (\partial'_j x^i = \frac{\partial x^i}{\partial x'^j})$

$\mathbf{S} = (s^i{}_j) = (\partial'_j x^i) = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$

b)

$\mathbf{g}' = \mathbf{S}^T\mathbf{S}$ (oder $g'_{ij} = \mathbf{e}'_i\cdot\mathbf{e}'_j$) $\rightarrow \mathbf{g}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$

$\det(\mathbf{g}') = r^4\sin^2\theta \rightarrow \sqrt{\det(\mathbf{g}')} = r^2\sin\theta.$
$\det(\mathbf{g}') = \det(\mathbf{S}^T\mathbf{S}) = \det(\mathbf{S}^T)\det(\mathbf{S}) = [\det(\mathbf{S})]^2 \rightarrow |\det(\mathbf{S})| = \sqrt{\det(\mathbf{g}')}$
oder
$\det(\mathbf{S}) = r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi + r^2\sin^3\theta\cos^2\phi = r^2\sin^3\theta + r^2\sin\theta\cos^2\theta =$
$r^2\sin\theta = \sqrt{\det(\mathbf{g}')}$

c) $dV = |dx'^1dx'^2dx'^3\mathbf{e}'_1\cdot(\mathbf{e}'_2\times\mathbf{e}'_3)|$
(oder $= |dx'^1dx'^2dx'^3\mathbf{e}'_i\cdot(\mathbf{e}'_j\times\mathbf{e}'_k$ wobei i,j,k einer Permutation von 1,2,3 ist.)
$= dx^1dx^2dx^3|\det(\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)||\det(\mathbf{S})| = r^2\sin\theta dr d\theta d\phi$
oder
$dV = |dx'^1dx'^2dx'^3\mathbf{e}'_1\cdot(\mathbf{e}'_2\times\mathbf{e}'_3)|$
$\mathbf{e}'_2\times\mathbf{e}'_3 = r^2\sin^2\theta\cos\phi\mathbf{e}_1 + r^2\sin^2\theta\sin\phi\mathbf{e}_2 + r^2\sin\theta\cos\theta\mathbf{e}_3$
$\rightarrow \mathbf{e}'_1\cdot(\mathbf{e}'_2\times\mathbf{e}'_3) = r^2\sin^3\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta = r^2\sin^3\theta + r^2\sin\theta\cos^2\theta = r^2\sin\theta$

d)
$\nabla\cdot\mathbf{w} = \mathbf{e}'^i\partial'_i\cdot w'^j\mathbf{e}'_j = \mathbf{e}'^i\partial'_i\cdot(Gw'^jG^{-1}\mathbf{e}'_j) = (\partial'_iGw'^j)\mathbf{e}'^i\cdot G^{-1}\mathbf{e}'_j + Gw'^j\mathbf{e}'^i\cdot(\partial'_iG^{-1}\mathbf{e}'_j)$
$= G^{-1}(\partial'_iGw'^j)\mathbf{e}'^i\cdot\mathbf{e}'_j = G^{-1}(\partial'_iGw'^j)\delta^i_j = G^{-1}(\partial'_iGw'^i)$

e)
$\mathbf{w} = w'^i\mathbf{e}'_i$, $G = r^2\sin\theta$
$\nabla\cdot\mathbf{w} = \frac{1}{G}\partial'_i(Gw'^i) = \frac{1}{G}(\partial_r(r^4\sin\theta\cos^2\phi) + \partial_\phi(r^4\sin\theta\sin^2\phi)) = 4r\cos^2\phi + 2r^2\sin\phi\cos\phi$

$\int_V (\nabla \cdot \mathbf{w}) dV = \int_0^2 dr \int_0^\pi d\theta \int_0^{2\pi} d\phi r^2 \sin\theta (4r \cos^2\phi + 2r^2 \sin\phi \cos\phi) = 4\pi \int_0^2 dr \int_0^\pi d\theta r^3 \sin\theta = 8\pi \int_0^2 dr r^3 = 32\pi^2$

Alternative Lösung:

Gaußscher Integralsatz $\int_V (\nabla \cdot \mathbf{w}) dV = \oint_F \mathbf{w} \cdot d\mathbf{F}$

$d\mathbf{F} = d\theta d\phi \mathbf{e}'_2 \times \mathbf{e}'_3 = r^2 \sin\theta \mathbf{e}'^1 d\theta d\phi$

$\oint_F \mathbf{w} \cdot d\mathbf{F} = \int_0^\pi d\theta \int_0^{2\pi} d\phi 2^4 \sin\theta \cos^2\phi = 16\pi \int_0^\pi d\theta \sin\theta = 32\pi^2$