5. Tutorium - Lösungen **22.11.2019**

- ANMERKUNG: Es liegt in der Verantwortung des Einzelnen, sich die Beispiele zunächst alleine und ganz ohne Hilfsmittel anzuschauen. Google, Wolfram Alpha, Lösungssammlungen, etc. helfen nur kurzfristig - leider nicht beim Test!

5.1 Differentialoperatoren (II)

a) $\nabla \times (\nabla x^i) \to \underbrace{\varepsilon_{jkl}}_{\text{asymmetrisch}} \underbrace{\partial_k \partial_l}_{\text{symmetrisch}} x^i = 0$

b) $\nabla x^i = \mathbf{f}^j \partial_j x^i = \mathbf{f}^j \delta_j{}^i = \mathbf{f}^i$

c) $V = \mathbf{f}_1 \cdot (\mathbf{f}_2 \times \mathbf{f}_3) = \sqrt{\det(\mathbf{g})} \to \mathbf{f}^1 = \frac{1}{V}\mathbf{f}_2 \times \mathbf{f}_3$ und $\mathbf{f}_1 = V\,\mathbf{f}^2 \times \mathbf{f}^3$ (siehe Bsp.4.2)

$\nabla \cdot (\frac{1}{V}\mathbf{f}_1) = \nabla \cdot (\mathbf{f}^2 \times \mathbf{f}^3) = (\nabla \times \underbrace{\mathbf{f}^2}_{=\nabla x^2 \text{ (Bsp.b)}}) \cdot \mathbf{f}^3 - \mathbf{f}^2 \cdot (\nabla \times \underbrace{\mathbf{f}^3}_{=\nabla x^3}) = (\underbrace{\nabla \times \nabla x^2}_{=0 \text{ (Bsp.a)}}) \cdot \mathbf{f}^3 - \mathbf{f}^2 \cdot (\underbrace{\nabla \times \nabla x^3}_{=0}) = 0$

Auf die gleiche Weise $\nabla \cdot (V^{-1}\mathbf{f}_2) = 0$ und $\nabla \cdot (V^{-1}\mathbf{f}_3) = 0$

Beweis des Hinweises

Da die Identität $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = (\nabla \times \mathbf{a}) \cdot \mathbf{b} - \mathbf{a} \cdot (\nabla \times \mathbf{b})$ unabhängig von der Basis ist, wird sie in der kartesischen Basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (d.h. $\mathbf{a} = a^i \mathbf{e}_i$ und $\mathbf{b} = b^i \mathbf{e}_i$) bewiesen.

$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \nabla \cdot (a^i \mathbf{e}_i \times b^j \mathbf{e}_j) = \nabla \cdot (a^i b^j \mathbf{e}_i \times \mathbf{e}_j) = \nabla \cdot (a^i b^j \varepsilon_{ijk} \mathbf{e}^k) = \mathbf{e}_\ell \partial^\ell \cdot (a^i b^j \varepsilon_{ijk} \mathbf{e}^k) = \varepsilon_{ijk} \mathbf{e}_\ell \cdot \mathbf{e}^k \partial^\ell (a^i b^j)$
$= \varepsilon_{ijk} \delta^k_\ell \partial^\ell (a^i b^j) = \varepsilon_{ijk}((\partial^k a^i) b^j + a^i (\partial^k b^j)) = (\nabla \times \mathbf{a})_j b^j - a^i (\nabla \times \mathbf{b})_i = (\nabla \times \mathbf{a}) \cdot \mathbf{b} - \mathbf{a} \cdot (\nabla \times \mathbf{b})$

d) $\mathbf{v} = v^i \mathbf{f}_i$

$$\nabla \cdot \mathbf{v} = \mathbf{f}^j \cdot \partial_j \left(v^i \mathbf{f}_i\right) = \mathbf{f}^j \cdot \partial_j \left(V v^i \tfrac{1}{V} \mathbf{f}_i\right) = \mathbf{f}^j \cdot \partial_j \left(V v^i\right) \tfrac{1}{V} \mathbf{f}_i + V v^i \underbrace{\mathbf{f}^j \cdot \partial_j \left(\frac{1}{V} \mathbf{f}_i\right)}_{=0 \text{(Bsp.c)}} = \tfrac{1}{V} \partial_j \left(V v^i\right) \delta^j{}_i = \tfrac{1}{V} \partial_i \left(V v^i\right)$$

Anmerkung : Laplace-Operator

$\nabla \cdot (\nabla \psi(\mathbf{x})) = \nabla \cdot (\mathbf{f}_i \partial^i \psi(\mathbf{x})) = \frac{1}{V} \partial_i (V \partial^i \psi(\mathbf{x})) = \frac{1}{V} \partial_i (V g^{ij} \partial_j \psi(\mathbf{x}))$

5.2 Lokale Transformation (II)

a) $\mathbf{e}'_j = s^i{}_j \mathbf{e}_i = \frac{\partial}{\partial x'^j} x^i \mathbf{e}_i$ (siehe Bsp.4.4), $\mathbf{S} = (\frac{\partial}{\partial x'^j} x^i) = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$

b) $\partial'_i = \frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j} = s^j{}_i \partial_j$

$\to \partial_r = \sin\theta\cos\phi\partial_x + \sin\theta\sin\phi\partial_y + \cos\theta\partial_z = (x^2 + y^2 + z^2)^{-1/2}(x\partial_x + y\partial_y + z\partial_z)$

$\partial_\phi = -r\sin\theta\sin\phi\partial_x + r\sin\theta\cos\phi\partial_y = x\partial_y - y\partial_x$

Anmerkung :

In der Quantenmechanik, ist der Impulsoperator durch $p_j = -i\partial_j$ definiert (i : die imaginäre Einheit).

$\to -i\partial_r = r^{-1}\mathbf{r} \cdot \mathbf{p} = p_r$ und $-i\partial_\phi = xp_y - yp_x = L_z$

c) $g'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j$

$\to \mathbf{g}' = \begin{pmatrix} \mathbf{e}_1'^T \\ \mathbf{e}_2'^T \\ \mathbf{e}_3'^T \end{pmatrix} \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{pmatrix} = \mathbf{S}^T \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix}^T \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} \mathbf{S} = \mathbf{S}^T\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$

$\mathbf{g}^{*\prime} = \mathbf{g}'^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/(r^2\sin^2\theta) \end{pmatrix}$

Anmerkung : Da die metrischen Tensoren diagonal sind, sind die Kugelkoordinaten orthogonale (aber nicht-normierte) Koordinaten.

d) Oberfläche $F = \{(r,\theta,\phi) | r = 3, 0 \le \theta < \pi/2, 0 \le \phi < 2\pi\} \to$ Tangentialebene der Fläche $F : \alpha\mathbf{e}'_2 + \beta\mathbf{e}'_3$

$\to$ Das Flächenelement $d\mathbf{F} = dx'^2\mathbf{e}'_2 \times dx'^3\mathbf{e}'_3\big|_{r=3} = V|_{r=3}\,\mathbf{e}'^1 dx'^2 dx'^3$ mit $V = \sqrt{\det(\mathbf{g})} = r^2\sin\theta$

$\to d\mathbf{F} = 9\sin\theta \mathbf{e}'^1 dx'^2 dx'^3$
Vektorfeld : $\mathbf{w} = w'_i \mathbf{e}'^i = \sin\phi \mathbf{e}'^2 + \sin\theta \mathbf{e}'^3$
$$\nabla\times\mathbf{w} = \mathbf{e}'^j\partial'_j \times w'_i\mathbf{e}'^i = (\partial'_j w'_i)\mathbf{e}'^j\times\mathbf{e}'^i + w'_i \underbrace{\mathbf{e}'^j\partial'_j\times\mathbf{e}'^i}_{=\nabla\times\mathbf{e}'^i=0 \text{ Bsp.5.1ab}} = (\partial'_j w'_i)\varepsilon^{jik}\frac{1}{V}\mathbf{e}'_k = (\partial'_j w'_i)\varepsilon^{jik}\frac{1}{r^2\sin\theta}\mathbf{e}'_k$$
$\int_F \nabla\times\mathbf{w}\cdot d\mathbf{F} = \int_0^{2\pi}\int_0^{\pi/2}(\partial'_j w'_i)_{r=3}\varepsilon^{jik}\frac{1}{9\sin\theta}\mathbf{e}'_k\cdot 9\sin\theta\mathbf{e}'^1 dx'^2dx'^3 = \int_0^{2\pi}\int_0^{\pi/2}(\partial'_j w'^i)_{r=3}\varepsilon^{ji1}dx'^2dx'^3$
$= \int_0^{2\pi}\int_0^{\pi/2}(\partial'_2 w'^3)_{r=3}dx'^2dx'^3 - \int_0^{2\pi}\int_0^{\pi/2}(\partial'_3 w'^2)_{r=3}dx'^2dx'^3$
$= \int_0^{2\pi}\int_0^{\pi/2}\cos\theta d\theta d\phi - \int_0^{2\pi}\int_0^{\pi/2}\cos(\phi)d\theta d\phi = 2\pi$
Alternative Lösung : Satz von Stokes $\int_F \nabla\times\mathbf{w}\cdot d\mathbf{F} = \int_C \mathbf{w}d\mathbf{s}$ mit $C = \{(x,y,z)|r=3, \theta=\pi/2, 0\le\phi<2\pi\}$
$d\mathbf{s} = dx'^i\mathbf{e}'_i \to \int_C \mathbf{w}\cdot d\mathbf{s} = \int_0^{2\pi}(\sin(\phi)\mathbf{e}'^2 + \sin\theta\mathbf{e}'^3)_C\cdot\mathbf{e}'_3 d\phi = \int_0^{2\pi}\sin\theta|_{\theta=\pi/2}d\phi = 2\pi$

5.3 Tensoren

a) Für orthonormale Basen $g_{ij} = \delta_{ij} \to \mathbf{A} = a^{ij}|\mathbf{e}_i\rangle\langle\mathbf{e}_j| = a^{ij}g_{ik}g_{\ell j}|\mathbf{e}^k\rangle\langle\mathbf{e}^\ell| \equiv a_{k\ell}|\mathbf{e}^k\rangle\langle\mathbf{e}^\ell|$
$$\to a_{k\ell} = a^{ij}g_{ik}g_{\ell j} = a^{k\ell} \to (a_{k\ell}) = \begin{pmatrix} -3 & 2 & -2 \\ -2 & 2 & -1 \\ 2 & -2 & 1 \end{pmatrix}$$
b) $\mathbf{e}_j = \mathbf{f}_i t^i{}_j$
$\mathbf{A} = a^{ij}|\mathbf{e}_i\rangle\langle\mathbf{e}_j| = a^{ij}t^k{}_i t^\ell{}_j|\mathbf{f}_k\rangle\langle\mathbf{f}_\ell| \equiv a'^{k\ell}|\mathbf{f}_k\rangle\langle\mathbf{f}_\ell|$
$$\to (a'^{k\ell}) = \mathbf{T}(a^{ij})\mathbf{T}^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} -3 & 2 & -2 \\ -2 & 2 & -1 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & 5 \end{pmatrix}$$
c) $$\begin{pmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{pmatrix}\begin{pmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \end{pmatrix} = \mathbf{I} \to \begin{pmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \end{pmatrix}^{-1}$$
$$\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \end{pmatrix}\mathbf{T} \to \begin{pmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \end{pmatrix}^{-1} = \mathbf{T}\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix}^{-1} = \mathbf{T}\begin{pmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \end{pmatrix}$$
$\mathbf{f}^1 = \mathbf{e}^1 + \mathbf{e}^3$, $\mathbf{f}^2 = \mathbf{e}^2 + \mathbf{e}^3$, $\mathbf{f}^3 = -\mathbf{e}^1 + 2\mathbf{e}^2$
d) $V^* = \mathbf{f}^1\cdot(\mathbf{f}^2\times\mathbf{f}^3) = -1$
$\mathbf{f}_1 = V^{*-1}(\mathbf{f}^2\times\mathbf{f}^3) = 2\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{f}_2 = V^{*-1}(\mathbf{f}^3\times\mathbf{f}^1) = -2\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$, $\mathbf{f}_3 = V^{*-1}(\mathbf{f}^1\times\mathbf{f}^2) = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$,
$$\to \mathbf{S} = \mathbf{T}^{-1} = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ -1 & 2 & -1 \end{pmatrix}$$
$g'_{ij} = \mathbf{f}_i\cdot\mathbf{f}_j = s^k{}_i\mathbf{e}_k\cdot s^\ell{}_j\mathbf{e}_\ell = s^k{}_i s^\ell{}_j\delta_{k\ell} = s^k{}_i s^k{}_j = (\mathbf{S}^T\mathbf{S})_{ij}$
$$= \begin{pmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ -1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 6 & -7 & 4 \\ -7 & 9 & -5 \\ 4 & -5 & 3 \end{pmatrix}$$
$g'^{ij} = \mathbf{f}^i\cdot\mathbf{f}^j = t^i{}_k\mathbf{e}^k\cdot t^j{}_\ell\mathbf{e}^\ell = t^i{}_k t^j{}_\ell\delta^{k\ell} = t^i{}_k t^j{}_k = (\mathbf{T}\mathbf{T}^T)^{ij}$
$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$
Anmerkung : $\mathbf{g}'^*\mathbf{g}' = (\mathbf{T}\mathbf{T}^T)(\mathbf{S}^T\mathbf{S}) = \mathbf{T}\underbrace{\mathbf{T}^T\mathbf{S}^T}_{=1}\mathbf{S} = \mathbf{T}\mathbf{S} = 1$
e) $\mathbf{A} = a'^{ij}|\mathbf{f}_i\rangle\langle\mathbf{f}_j| = a'^{ij}g'_{\ell j}|\mathbf{f}_i\rangle\langle\mathbf{f}^\ell| = a'^i{}_\ell|\mathbf{f}_i\rangle\langle\mathbf{f}^\ell|$
$$(a'^i{}_\ell) = (a'^{ij})\mathbf{g}'^T = \begin{pmatrix} -2 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & 5 \end{pmatrix}\begin{pmatrix} 6 & -7 & 4 \\ -7 & 9 & -5 \\ 4 & -5 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$\mathbf{A} = a'^{ij}|\mathbf{f}_i\rangle\langle\mathbf{f}_j| = a'^{ij}g'_{ik}g'_{\ell j}|\mathbf{f}^k\rangle\langle\mathbf{f}^\ell| = a'^i{}_\ell g'_{ik}|\mathbf{f}^k\rangle\langle\mathbf{f}^\ell| = a'_{k\ell}|\mathbf{f}^k\rangle\langle\mathbf{f}^\ell|$
$$(a'_{k\ell}) = \mathbf{g}'^T(a'^i{}_\ell) = \begin{pmatrix} 6 & -7 & 4 \\ -7 & 9 & -5 \\ 4 & -5 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -6 & 0 & 4 \\ 7 & 0 & -5 \\ -4 & 0 & 3 \end{pmatrix}$$

f) $a^{ij}a_{ji} = \mathrm{Tr}\left[\begin{pmatrix} -3 & 2 & -2 \\ -2 & 2 & -1 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} -3 & 2 & -2 \\ -2 & 2 & -1 \\ 2 & -2 & 1 \end{pmatrix}\right] = \mathrm{Tr}\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{pmatrix} = 2$

$a'^{ij}a'_{ji} = \mathrm{Tr}\left[\begin{pmatrix} -2 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & 5 \end{pmatrix}\begin{pmatrix} -6 & 0 & 4 \\ 7 & 0 & -5 \\ -4 & 0 & 3 \end{pmatrix}\right] = \mathrm{Tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2$

Alternative Lösung :

$a'^{ij}a'_{ji} = \mathrm{Tr}(\mathbf{T}(a^{ij})\mathbf{T}^T\mathbf{S}^T(a_{k\ell})\mathbf{S}) = \mathrm{Tr}(\mathbf{T}(a^{ij}a_{j\ell})\mathbf{S}) = \mathrm{Tr}(\mathbf{ST}(a^{ij}a_{j\ell})) = \mathrm{Tr}((a^{ij}a_{j\ell})) = a^{ij}a_{ji}$