4. Tutorium - Lösungen **5.11.2021**

4.1 Differentialoperatoren

a) $\nabla \mathbf{x} = \mathbf{e}_i \partial_i x_j \mathbf{e}_j = \partial_i x_j \delta_{ij} = \partial_i x_i = \delta_{ii} = 3$
b) $\nabla x = \mathbf{e}_i \partial_i \sqrt{x_j \mathbf{e}_j x_k \mathbf{e}_k} = \mathbf{e}_i \partial_i \sqrt{x_j x_k \delta_{jk}} = \mathbf{e}_i \partial_i \sqrt{x_j x_j} = \mathbf{e}_i \frac{1}{2\sqrt{x_k x_k}} \partial_i (x_j x_j) = \mathbf{e}_i \frac{1}{2\sqrt{x_k x_k}} 2\delta_{ij} x_j = \mathbf{e}_i \frac{x_i}{\sqrt{x_k x_k}}$
c) $\nabla \times (\nabla \times \mathbf{A}) = \mathbf{e}_i \partial_i \times (\mathbf{e}_j \partial_j \times a_k \mathbf{e}_k) = \mathbf{e}_i \partial_i \times (\varepsilon_{\ell jk} \mathbf{e}_\ell \partial_j a_k) = \varepsilon_{mi\ell} \mathbf{e}_m \partial_i (\varepsilon_{\ell jk} \partial_j a_k)$
$= \varepsilon_{mi\ell} \varepsilon_{\ell jk} \mathbf{e_m} \partial_i \partial_j a_k = (\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}) \mathbf{e}_m \partial_i \partial_j a_k = \mathbf{e}_j \partial_i \partial_j a_i - \mathbf{e}_k \partial_i \partial_i a_k$
Anmerkung : Für orthonormale Rechtssysteme, gilt $\mathbf{e}_j \times \mathbf{e}_k = \varepsilon_{ijk} \mathbf{e}_i$ (z.B. $\mathbf{e}_y \times \mathbf{e}_z = \mathbf{e}_x$, $\mathbf{e}_x \times \mathbf{e}_z = -\mathbf{e}_y, \cdots$)
$\nabla(\nabla \cdot \mathbf{A}) = \mathbf{e}_i \partial_i (\mathbf{e}_j \partial_j a_k \mathbf{e_k}) = \mathbf{e}_i \partial_i \partial_j a_j$
$\mathbf{A} \cdot [\nabla \times (\nabla \times \mathbf{A}) - \nabla(\nabla \cdot \mathbf{A})] = a_n \mathbf{e}_n (\mathbf{e}_j \partial_i \partial_j a_i - \mathbf{e}_k \partial_i \partial_i a_k - \mathbf{e}_i \partial_i \partial_j a_j) = a_j \partial_i \partial_j a_i - a_k \partial_i \partial_i a_k - a_i \partial_i \partial_j a_j =$
$-a_k \partial_i \partial_i a_k = -\mathbf{A} \nabla^2 \mathbf{A}$
d) $\partial'^i = \frac{\partial}{\partial x'_i} = \frac{\partial x_j}{\partial x'_i} \frac{\partial}{\partial x_j} = \frac{\partial x_j}{\partial x'_i} \partial^j$
$x'_k = s^j{}_k x_j \to x_j = x'_k t^k{}_j \to \frac{\partial x_j}{\partial x'_i} = \delta^i_k t^k{}_j = t^i{}_j \to \partial'^i = t^i{}_j \partial^j$
$\partial'_i = \frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j} = \frac{\partial x^j}{\partial x'^i} \partial_j$
$x'^k = t^k{}_j x^j \to x^j = x'^k s^j{}_k \to \frac{\partial x^j}{\partial x'^i} = \delta^k_i s^j{}_k = s^j{}_i \to \partial'_i = s^j{}_i \partial_j$
Anmerkung : $\nabla = \mathbf{f}_i \partial'^i = \mathbf{f}^i \partial'_i$
e) $\nabla \mathbf{x} = \mathbf{f}^j \partial_j x'^i \mathbf{f}_i = \partial_j x'^i \delta^j_i = \partial_j x'^j = \delta^j_j = 3.$

4.2 Spektraltheorem

a) $\frac{d}{dt} x_i(t) = \mathbf{e}_i^T \frac{d}{dt} \mathbf{x}(t) = \mathbf{e}_i^T \mathbf{A} \mathbf{x}(t) = \mathbf{e}_i^T \mathbf{A} x_j(t) \mathbf{e}_j = a_{ij} x_j(t)$
b) Eigenwertgleichung : $a_{ij} x_j = \lambda x_i \to (a_{ij} - \lambda \delta_{ij}) x_j = 0$
Wenn $\det(a_{ij} - \lambda \delta_{ij}) = 0$, existiert nicht-triviale Lösungen

$$\det(a_{ij} - \lambda\delta_{ij}) = \det \begin{vmatrix} -\lambda & \sqrt{2} & 0 \\ \sqrt{2} & -\lambda & \sqrt{2} \\ 0 & \sqrt{2} & -\lambda \end{vmatrix} = -\lambda^3 + 4\lambda = \lambda(\lambda+2)(\lambda-2) = 0 \to \lambda_1 = 2,\ \lambda_2 = 0,\ \lambda_3 = -2$$

$$\text{Eigenvektoren : } \begin{pmatrix} -\lambda & \sqrt{2} & 0 \\ \sqrt{2} & -\lambda & \sqrt{2} \\ 0 & \sqrt{2} & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\lambda x + \sqrt{2} y \\ \sqrt{2} x - \lambda y + \sqrt{2} z \\ \sqrt{2} y - \lambda z \end{pmatrix} = 0$$

Wenn $\lambda = 2$, z.B. $(x, y, z) = (1, \sqrt{2}, 1)$. $\to \mathbf{e}'_1 = (1/2)(\mathbf{e}_1 + \sqrt{2}\mathbf{e}_2 + \mathbf{e}_3)$
Wenn $\lambda = 0$, z.B. $(x, y, z) = (1, 0, -1)$. $\to \mathbf{e}'_2 = (1/\sqrt{2})(\mathbf{e}_1 - \mathbf{e}_3)$
Wenn $\lambda = -2$, z.B. $(x, y, z) = (1, -\sqrt{2}, 1)$. $\to \mathbf{e}'_3 = (1/2)(\mathbf{e}_1 - \sqrt{2}\mathbf{e}_2 + \mathbf{e}_3)$

$$\text{oder } \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

Anmerkung : Für einen selbstadjungierter Operator sind die Eigenvektoren orthogonal, d.h. $\mathbf{e}'_i \mathbf{e}'_j \propto \delta_{ij}$. Die Transformationsmatrix $\mathbf{S}$ erfüllt die Bedingung $\det\mathbf{S} \neq 0$ und die Eigenvektoren sind linear unabhängig.
c) $e'_{ijk} = \mathbf{e}_j \mathbf{E}'_i \mathbf{e}_k = \mathbf{e}_j \mathbf{e}'_i \otimes \mathbf{e}'_i \mathbf{e}_k = (\mathbf{e}_j \mathbf{e}'_i)(\mathbf{e}'_i \mathbf{e}_k)$ (ohne Summe über i)

$$\text{Wenn } i = 1,\ (e'_{1jk}) = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$$

$$\text{Wenn } i = 2,\ (e'_{2jk}) = \frac{1}{2} \begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix} \frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & -\sqrt{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\text{Wenn } i = 3,\ (e'_{3jk}) = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{2} & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$\lambda_k e_{kij} = 2e_{1ij} + 0\, e_{2ij} - 2e_{3ij} \to (\lambda_k e_{kij}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = (a_{ij})$

d) In der Eigenbasis $\mathbf{A} = \sum_{i,j} a'_{ij} \mathbf{e}'_i \otimes \mathbf{e}'_j = \sum_i \lambda_i \mathbf{E}'_i = \sum_i \lambda_i \mathbf{e}'_i \otimes \mathbf{e}'_i$

$\to a'_{ij} = \lambda_i$ wenn $i = j$, sonst $a'_{ij} = 0$ (diagonale Matrix)

Alternativ :

$\mathbf{A}\mathbf{e}'_j = \lambda_j \mathbf{e}'_j \to a'_{ij} = \mathbf{e}'_i \mathbf{A} \mathbf{e}'_j = \mathbf{e}'_i \lambda_j \mathbf{e}'_j = \lambda_j \delta_{ij}$ (ohne Summe über j)

e) Auf die gleiche Weise wie (a)

$\frac{d}{dt} x'_i(t) = a'_{ij} x'_j(t) \to \frac{d}{dt} x'_i(t) = \lambda_i x'_i(t)$ (ohne Summe über i) $\to x'_i(t) = e^{\lambda_i t} x'_i(0)$.

f) $\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \sum_{i,j} \lambda_i \mathbf{E}'_i \lambda_j \mathbf{E}'_j = \sum_{i,j} \lambda_i \lambda_j \mathbf{e}'_i \otimes \mathbf{e}'_i \mathbf{e}'_j \otimes \mathbf{e}'_j = \sum_{i,j} \lambda_i \lambda_j \mathbf{e}'_i \otimes \delta_{ij} \otimes \mathbf{e}'_j = \sum_i \lambda_i \lambda_i \mathbf{e}'_i \otimes \mathbf{e}'_i = \sum_i (\lambda_i)^2 \mathbf{E}'_i$

$\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \sum_i (\lambda_i)^3 \mathbf{E}'_i$

$\vdots$

$\mathbf{H} = \mathbf{A}^n = \mathbf{A}\mathbf{A}^{n-1} = \sum_i (\lambda_i)^n \mathbf{E}'_i$

In der Basis $\mathcal{B}'$, $(h'_{ij}) = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (-2)^n \end{pmatrix}$

und in der Basis $\mathcal{B}$, $(h_{ij}) = 2^{n-2} \begin{pmatrix} 1 + (-1)^n & \sqrt{2}(1 - (-1)^n) & 1 + (-1)^n \\ \sqrt{2}(1 - (-1)^n) & 2(1 + (-1)^n) & \sqrt{2}(1 - (-1)^n) \\ 1 + (-1)^n & \sqrt{2}(1 - (-1)^n) & 1 + (-1)^n \end{pmatrix}$

g) $[\mathbf{H}, \mathbf{A}] = (\lambda_i)^n \mathbf{E}'_i \lambda_j \mathbf{E}'_j - \lambda_j \mathbf{E}'_j (\lambda_i)^n \mathbf{E}'_i - = (\lambda_i)^{n+1} \mathbf{E}'_i - (\lambda_i)^{n+1} \mathbf{E}'_i = 0$

oder $[\mathbf{H}, \mathbf{A}] = [\mathbf{A}^n, \mathbf{A}] = \mathbf{A}^{n+1} - \mathbf{A}^{n+1} = 0$

Anmerkung : Wenn $[\mathbf{A}, \mathbf{B}] = 0$ und $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$,

$\mathbf{A}(\mathbf{B}\mathbf{x}) = \mathbf{B}\mathbf{A}\mathbf{x} = \lambda(\mathbf{B}\mathbf{x}) \to \mathbf{B}\mathbf{x}$ ist ein Eigenvektor des Operators $\mathbf{A}$ mit dem Eigenwert λ.

Ohne Entartung $\mathbf{B}\mathbf{x} = \alpha \mathbf{x}$. ($\mathbf{x}$ ist auch ein Eigenvektor des Operators $\mathbf{B}$)

h) $\sum_n \mathbf{A}^n t^n / n! = \sum_n \sum_i (\lambda_i t)^n / n! \mathbf{E}'_i = \sum_i e^{\lambda_i t} \mathbf{E}'_i = e^{\mathbf{A} t}$

$\mathbf{x}(t) = \sum_i x'_i(t) \mathbf{e}'_i = \sum_i e^{\lambda_i t} x'_i(0) \mathbf{e}'_i$

$e^{\mathbf{A} t} \mathbf{x}(0) = \sum_{i,j} e^{\lambda_i t} \mathbf{E}'_i x'_j(0) \mathbf{e}'_j = \sum_{i,j} e^{\lambda_i t} x'_j(0) \delta_{ij} \mathbf{e}'_i = \sum_i e^{\lambda_i t} x'_i(0) \mathbf{e}'_i$

$\mathbf{x}(t) = e^{\mathbf{A} t} \mathbf{x}(0)$