1 Ehrenfesttheorem

For analytic Potentials we can derive the following Formula:

$$\begin{aligned}
[\hat{p}, V(\hat{x})] &= [\hat{p}, \sum_{n=1}^{\infty} v_n (\hat{x})^n] \\
&= \sum_{n=1}^{\infty} v_n [\hat{p}, (\hat{x})^n] = \sum_{n=1}^{\infty} v_n n (\hat{x})^{n-1}(-i\hbar) = -i\hbar \frac{dV(\hat{x})}{d\hat{x}}
\end{aligned}$$

We can use this result to calculate the time development of the mean values of $\hat{p}, \hat{x}$.

$$\begin{aligned}
i\hbar \frac{\hat{p}}{dt} = [\hat{p}, \frac{(\hat{p})^2}{2m} + \lambda(\hat{x})^n] = [\hat{p}, \lambda(\hat{x})^n] &= -i\hbar n(\hat{x})^{n-1} \\
\Rightarrow \frac{d\hat{p}}{dt} &= -n(\hat{x})^{n-1} \qquad (1) \\
\frac{d\hat{x}}{dt} = [\hat{x}, \frac{(\hat{p})^2}{2m} + \lambda(\hat{x})^n] = [\hat{x}, \frac{(\hat{p})^2}{2m}] = \frac{1}{2m}[\hat{x}, (\hat{p})^2] &= i\hbar \frac{\hat{p}}{m}
\end{aligned}$$

For all n we have:

$$\frac{d < \hat{x} >}{dt} = i\hbar \frac{< \hat{p} >}{m}$$

For n=0:

$$\frac{d < \hat{p} >}{dt} = 0$$

For n=2:

$$\frac{d < \hat{p} >}{dt} = -2\lambda < \hat{x} >$$

In general for $0 \leq n \leq 2$ we can write:

$$< \frac{dV(\hat{x})}{d\hat{x}} > = \frac{dV(< \hat{x} >)}{d < \hat{x} >}$$

This means that the mean values of $\hat{p}, \hat{x}$ follow the classical equation of motion.(Ehrenfest theorem)

2 Virialsatz der QM

The quantum mechanical operator corresponding to the classical observable xp(virial) is not $\hat{x}\hat{p}$ since this operator is not hermitian but $\frac{1}{2}(\hat{x}\hat{p}+\hat{p}\hat{x})$. This procedure is called Weyl symmetrization.

$$\begin{aligned}
\hat{v} &= \frac{1}{2}(\hat{x}\hat{p}+\hat{p}\hat{x}) \\
i\hbar\frac{d<\hat{v}>}{dt} &=< [\hat{v},H] >=< \hat{v}H > - < H\hat{v} >= E(< \hat{v} > - < \hat{v} >) = 0 \\
i\hbar\frac{d\hat{v}}{dt} &= \frac{1}{2}[\hat{x}\hat{p}+\hat{p}\hat{x},\frac{(\hat{p})^2}{2m}+\lambda(\hat{x})^n] \\
&= \frac{1}{2}([\hat{x}\hat{p},\frac{(\hat{p})^2}{2m}]+[\hat{x}\hat{p},\frac{(\hat{p})^2}{2m}]+[\hat{p}\hat{x},\lambda(\hat{x})^n]]+[\hat{p}\hat{x},\lambda(\hat{x})^n]]) \\
&=\frac{1}{2}(\hat{x}[\hat{p},\frac{(\hat{p})^2}{2m}]+[\hat{x},\frac{(\hat{p})^2}{2m}]\hat{p}+\hat{p}[\hat{x},\frac{(\hat{p})^2}{2m}]+[\hat{p},\frac{(\hat{p})^2}{2m}]\hat{x}+ \\
&\quad \hat{x}[\hat{p},\lambda(\hat{x})^n]+[\hat{x},\lambda(\hat{x})^n]\hat{p}+\hat{p}[\hat{x},\lambda(\hat{x})^n]+[\hat{p},\lambda(\hat{x})^n]\hat{x}) \\
&=\frac{1}{2}([\hat{x},\frac{(\hat{p})^2}{2m}]\hat{p}+\hat{p}[\hat{x},\frac{(\hat{p})^2}{2m}]+\hat{x}[\hat{p},\lambda(\hat{x})^n]+[\hat{p},\lambda(\hat{x})^n]\hat{x}) \\
&= i\hbar\frac{(\hat{p})^2}{2m} - i\hbar\hat{x}n\lambda(\hat{x})^{n-1} \\
&\Rightarrow\frac{d<\hat{v}>}{dt} =< \frac{(\hat{p})^2}{m} > -n\lambda < (\hat{x})^n >= 2 < T > -n < V >
\end{aligned}$$

For $< \hat{p}^2 >$ and $n = 2$ we get:

$$\begin{aligned}
< E >&=< T > + < V > \\
2 < T >&= 2 < V > \\
< E >= 2 < T >&=< \hat{p}^2 > /m \Rightarrow< \hat{p}^2 >=< E > m
\end{aligned}$$

3 Time development and measurement

The eigenvalues of the observable $\hat{O}$ are $o = \pm\frac{1}{2}$ these are the possible values that can be measured for the observable $\hat{O}$. To calculate the time development of the system we first calculate the eigenbasis of the Hamiltonian:

$$H|+g>= g|+g> \qquad H|-g>= -g|-g>$$
$$|+g>= \frac{1}{\sqrt{2}}(|1>-|2>)$$
$$|-g>= \frac{1}{\sqrt{2}}(|1>+|2>)$$

The state at t=0 is $|\psi(t=0)>=|1>$. To get the time development we rewrite this state in the eigenbasis of the Hamiltonian:

$$|1> = \frac{1}{\sqrt{2}}(|+g>+|-g>)$$
$$|\psi(t)> = \frac{1}{\sqrt{2}}(e^{-igt/\hbar}|+g>+e^{-igt/\hbar}|-g>)$$
$$= cos(gt/\hbar)|1>+isin(gt/\hbar)|2>$$

The Possibilities for the measurement of $o = \pm\frac{1}{2}$ are given by:

$$P(o = \frac{1}{2})(t) = |<1|\psi(t)>|^2 = cos(gt/\hbar)^2$$
$$P(o = -\frac{1}{2})(t) = |<2|\psi(t)>|^2 = sin(gt/\hbar)^2$$

If on the other hand the initial state would have been $|2>$ the time development would be:

$$|\psi_2(t)> = \frac{1}{\sqrt{2}}(e^{-igt/\hbar}|+g>-e^{-igt/\hbar}|-g>)$$
$$= -isin(gt/\hbar)|1>-cos(gt/\hbar)|2>$$

Now we consider the following situation: At $t = 0$ the system is in the state $|1>$ then at $t = t^*/2$ we measure $\hat{O}$. Depending on the result the system is either in the state $|1>$ or $|2>$. Afterwards we let the system evolve and at $t = t^*$ we measure $\hat{O}$ again.

The Probability to find $o = \frac{1}{2}$ at $t = t^*/2$ and $o = \frac{1}{2}$ at $t = t^*$ is denoted by $P(\frac{1}{2}, \frac{1}{2})$. The Probability to find $o = \frac{1}{2}$ at $t = t^*/2$ and $o = -\frac{1}{2}$ at $t = t^*$ by $P(\frac{1}{2}, -\frac{1}{2})$ and so forth.

$$
\begin{aligned}
P(\frac{1}{2}, \frac{1}{2}) &= Cos^4(gt^*/(2\hbar)) \\
P(\frac{1}{2}, -\frac{1}{2}) &= Cos^2(gt^*/(2\hbar))Sin^2(gt^*/(2\hbar)) \\
P(-\frac{1}{2}, \frac{1}{2}) &= Sin^4(gt^*/(2\hbar)) \\
P(-\frac{1}{2}, -\frac{1}{2}) &= Cos^2(gt^*/(2\hbar))Sin^2(gt^*/(2\hbar))
\end{aligned}
$$

The total probability to find the value $o = \frac{1}{2}$ at $t = t^*$ if we measure the observable at $t = t^*/2$ is:

$$\tilde{P}(o = \frac{1}{2}) = P(\frac{1}{2}, \frac{1}{2}) + P(-\frac{1}{2}, \frac{1}{2}) = Cos^4(gt^*/(2\hbar)) + Sin^4(gt^*/(2\hbar)) \tag{2}$$

which is different from the result we got without measurement at $t = t^*/2$.

If on the other hand we look at $|\psi(t)$ and ask for the probability to find the energy $\pm g$ at time $t = t^*$. We find that $P(E = +g) = 1/2$ and $P(E = -g) = 1/2$ for any time. If however we measure the energy once we know that the system stays in that eigenfunction for all time.(This is only true since the Hamilton is time independent)