Technische Universität Wien

Prof. Dr. K. Held
Dr. G. Sangiovanni, Dr. A. Toschi Dr. A. Toth, Dr. G. Rohringer

9. Übung zur Quantenmechanik I, Lösungen

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17. Spin und Heisenbergsche Unschärfe-Relation

5 Punkte

a) Commutation relations
As can easily be verified using the explicit definition of the Pauli matrices:

$$\sigma_i \sigma_j = \epsilon_{ijk} \sigma_k \tag{1}$$

from this one can derive the commutation relation for the S_i matrices.

b) Expectation values

The eigenstates of S_z are $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, with eigenvalues $\pm\hbar/2$. The expectation values $\langle S_i \rangle$, for $i = x$ or $i = y$, are then:

$$\frac{\hbar}{2} \begin{pmatrix} 1 & 0 \end{pmatrix} \sigma_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \tag{2}$$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \end{pmatrix} \sigma_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \tag{3}$$

while, $\langle + | S_z | + \rangle = +\hbar/2$ and $\langle - | S_z | - \rangle = -\hbar/2$. This is a characteristic of the eigenstates: if one measures an observable on its own eigenstate the outcome of the measure is the corresponding eigenvalue with probability one, thus the eigenvalue is also the expectation value of the measure. Since $\sigma_i^2 = \mathbf{1}$ one always has $\langle S_i^2 \rangle = \frac{\hbar^2}{4}$. Thus $\Delta S_x = \Delta S_y = \hbar/2$, while $\Delta S_z = 0$: there is no uncertanty in the measure of an observable if the measure is taken on an eigenstate of the same observable, as stated above.

c) Uncertainty relation
We need to check the general relation:

$$\Delta_A \Delta_B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle| \tag{4}$$

for $\hat{A}$ and $\hat{B}$ being the spin operators and Δ_A and Δ_B the relative uncerainties. This is easily done using the commutation relations of point a of this exercise, and the expectation values and uncertainties of point b. For example one has $\Delta S_x \Delta S_y = \hbar^2/4$, while at the second hand of Eq. 4 one gets the following:

$$\frac{1}{2} \langle [S_x, S_y] \rangle = 1/2 \hbar \langle S_z \rangle = \hbar^2/4, \tag{5}$$

and the uncertainty relation is verified.

d) $S_\pm$ operators

The operators that increase or decrease the spin in the z direction are:

$$S_\pm = \frac{1}{2}\left(S_x \pm iS_y\right). \tag{6}$$

Using the explicit definition of the Pauli matrices one can easily check the following relations:

$$\begin{aligned} S_+|-\rangle &= \hbar|+\rangle && (7)\\ S_-|+\rangle &= \hbar|-\rangle && (8)\\ S_+|+\rangle = S_-|-\rangle &= 0, && (9) \end{aligned}$$

Using the commutation relations derived in point a) of this exercise one can derive:

$$[S_z, S_\pm] = \pm\hbar S_\pm. \tag{10}$$

This relation can also be used to show that $S_\pm$ increase or decrease the spin along z. For doing this we can suppose to start from an eigenstate of S_z, for example $|+\rangle$, then apply the operator S_z and check if we got an eigenstate and with which eigenvalue:

$$S_z(S_-|+\rangle) = S_-S_z|+\rangle + [S_z, S_-]|+\rangle = \hbar/2 S_-|+\rangle - \hbar S_-|+\rangle = -\hbar/2 S_-|+\rangle, \tag{11}$$

that shows that $S_-|+\rangle$ is eigenstate of the S_z with eigenvalue $-\hbar/2$. In the derivation were used only the commutation relations of the operators and not their explicit matrix form, that in some cases is not available or not of practical use. Finally the last commutator is:

$$[S_+, S_-] = \hbar/2 S_z. \tag{12}$$

18. Darstellung des Drehimpulsoperators *5 Punkte*

a) Eigenvectors of L_x

Let us introduce the following operators:

$$L_\pm = L_x \pm iL_y. \tag{13}$$

These operators increase (L_+) or decrease (L_-) the angular momentum in the z direction, as one can check from the commutation relations in complete analogy to the spin case treated in the previous exercise, in particular it can be shown that:

$$\begin{aligned} L_+|l,m\rangle &= \hbar\sqrt{l(l+1) - m(m+1)}|l, m+1\rangle && (14)\\ L_-|l,m\rangle &= \hbar\sqrt{l(l+1) - m(m-1)}|l, m-1\rangle && (15) \end{aligned}$$

The L_x operator can be expressed in terms of $L_\pm$ and this can be used for constructing the matrix elements of L_x for the case $l = 1$:

$$L_x^{m,m'} = \langle l = 1, m|\frac{1}{2}(L_+ + L_-)|l = 1, m'\rangle, \tag{16}$$

and then:

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{17}$$

whose eigenvalues are $|\pm_x\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \pm\sqrt{2} \\ 1 \end{pmatrix}$ with eigenvalues $\lambda_\pm = \pm\hbar$, and $|0_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ with eigenvalue $\lambda_0 = 0$.

b) Eigenvalues of L_x

The spectrum of L_x is the same of L_z, as one should expect.

c) Measures of L_z and L_x

After measuring L_z the system is in an eigenstate of L_z denoted by $|l = 1, m\rangle$. The probability of measuring $\hbar\lambda$ along the x direction is given by:

$$|\langle\lambda_x|l = 1, m\rangle|^2, \tag{18}$$

with $\lambda = \pm, 0$.

Calling $P(\lambda_x, \lambda_z)$ the probability of measuring λ_x along x after having measured λ_z along z one has:

$$\begin{gathered}
P(+,+) = 1/4 \\
P(0,+) = 1/2 \\
P(-,+) = 1/4 \\
P(+,0) = 1/2 \\
P(0,0) = 0 \\
P(-,0) = 1/2 \\
P(+,-) = 1/4 \\
P(0,-) = 1/2 \\
P(-,-) = 1/4.
\end{gathered} \tag{19}$$