5. Tutorium - Quantentheorie I - 23.11.2012 - Lösung

1)a)

$$\left(\hat{F}(\hat{A})\right)^\dagger = \left(\sum_{n=0}^{\infty} f_n\ \hat{A}^\dagger\right)^\dagger = \left[\sum_{n=0}^{\infty}\ f_n \cdot \left(\Pi_{i=1}^{n} \hat{A}\right)\right]^\dagger = \sum_{n=0}^{\infty}\ \Pi_{i=1}^{n} \underbrace{\left(\hat{A}\right)^\dagger}_{\hat{A}^\dagger = \hat{A}} = \sum_{n=0}^{\infty}\ \Pi_{i=1}^{n} \hat{A} = \hat{F}\left(\hat{A}\right)$$

b)

$$\hat{F}(\hat{A})\left|\psi_n\right\rangle = a\left|\psi_n\right\rangle$$

$$\sum_{n=0}^{\infty}\ f_n\ (\hat{A})^n \left|\psi\right\rangle = \sum_{n=0}^{\infty}\ f_n\Big(\sum_n \underbrace{(a_n)}_{EW}{}^n \left|\psi_n\right\rangle \underbrace{\left\langle\psi_n\right|\Big)\left|\psi_n\right\rangle}_{=1} = \sum_{n=0}^{\infty}\ f_n\ (a_n)^n \left|\psi_n\right\rangle$$

$$(\hat{A})^n = \sum_i\ (a_i)^n$$

Beweis:

$$\text{Spektraldarstellung: } (\hat{A})^m = \sum_i (a_i)^m \left|a_i\right\rangle\left\langle a_i\right| \xrightarrow{m=0} \hat{A} = \underbrace{(a_i)^0}_{=1}\left|a_0\right\rangle\left\langle a_0\right| = \mathbb{1}$$

$$\text{Induktionsannahme: } (\hat{A})^{M+1} = (\hat{A})^M \cdot \hat{A} = \sum_j\ (a_j)^M \left|a_j\right\rangle\ \underbrace{\left\langle a_j\right| \cdot \hat{A}}_{\langle a_j | a_j{}^* = \langle a_j | a_j}\ = \sum_j (a_j)^{M+1} \cdot \left|a_j\right\rangle\left\langle a_j\right|$$

EV von $\hat{A} \neq$ EV von $\hat{F}(\hat{A})$

c)

$$\hat{F}(\hat{A}) = \sum_n f(\hat{A})^n = \sum_n\ \underbrace{\left|a_n\right\rangle\left\langle a_n\right|}_{=\mathbb{1}}\ f_n \underbrace{\cdot (a_n)^n \left|a_n\right\rangle\left\langle a_n\right|}_{=(\hat{A})^n} =$$

$$= \sum_n \left|a_n\right\rangle f_n\ (a_n)^n\ \left\langle a_n | a_n\right\rangle\left\langle a_n\right| = \sum_n \left|a_n\right\rangle f_n\ (a_n)^n \left\langle a_n\right|$$

Spektraldarstellung = Darstellung in der Eigenbasis:

$$\hat{A} = \sum_i a_i \left|a_i\right\rangle\left\langle a_i\right| \qquad a_i \ldots EW \qquad \left|a_i\right\rangle \text{ und } \left\langle a_i\right| \ldots EV$$

d) Operator $\hat{S}$ bez. bel. Basis $\{\langle e_1|, \langle e_2|\}$ $\quad\big(\{e\}$-Darstellung$\big)$

$$\hat{S}^{\{e\}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Ges: Operator $B = e^{\hat{S}}$ in $\{e\}$-Darstellung

EW, EV von $\hat{S}$:

$$\lambda_1 = -1 = a_1 \qquad v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} = |\psi_1\rangle$$

$$\lambda_2 = 1 = a_2 \qquad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} = |\psi_2\rangle$$

$$e^{\hat{S}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\hat{S}\right)^n = \sum_{n=0}^{\infty} \left(\sum_i \left((a_i)^n |\psi_i\rangle \langle\psi_i| \right) \right) = \sum_i \sum_n \frac{1}{n!} (\underbrace{a_i}_{EW})^n \cdot \underbrace{|\psi_n\rangle \langle\psi_n|}_{EV} = \sum_i e^{a_i} \cdot |\psi_i\rangle \langle\psi_i| =$$

$$= \underbrace{e^{-1}}_{e^{a_1}} \cdot \underbrace{\frac{1}{2} \begin{pmatrix} i \\ 1 \end{pmatrix} \begin{pmatrix} -i & 1 \end{pmatrix}}_{|\psi_1\rangle\langle\psi_1|} + \underbrace{e^{1}}_{e^{a_2}} \cdot \underbrace{\frac{1}{2} \begin{pmatrix} -i \\ 1 \end{pmatrix} \begin{pmatrix} i & 1 \end{pmatrix}}_{|\psi_1\rangle\langle\psi_1|} = \frac{1}{2e} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} + \frac{e}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \begin{pmatrix} cosh(1) & -i\ sinh(1) \\ i\ sinh(1) & cosh(1) \end{pmatrix}$$

2)Es gilt:

$$\hat{A}\,|e_1\rangle = a\,|e_1\rangle \qquad \hat{B}\,|e_1\rangle = b\,|e_1\rangle$$
$$\hat{A}\,|e_2\rangle = -a\,|e_2\rangle \qquad \hat{B}\,|e_2\rangle = -ib\,|e_3\rangle$$
$$\hat{A}\,|e_3\rangle = -a\,|e_3\rangle \qquad \hat{B}\,|e_3\rangle = ib\,|e_2\rangle$$

a)Ges: Matrix $\hat{A}^{\{e\}}$ und $\hat{b}^{\{e\}}$ $\longrightarrow$ Operatoren A und B ind Basis $\{|e_1\rangle\,, |e_2\rangle\,, |e_3\rangle\}$

Es gilt:

$$(\hat{A}^{\{e\}})_{jk} = \langle e_j|\,\hat{A}\,|e_k\rangle$$

$$\langle e_1|\,\hat{A}\,|e_1\rangle = \langle e_1|\,a\,|e_1\rangle = a$$
$$\langle e_2|\,\hat{A}\,|e_2\rangle = -a$$
$$\langle e_3|\,\hat{A}\,|e_3\rangle = -a$$

alle anderen Einträge sind Null.

$$\longrightarrow \hat{A}^{\{e\}} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}$$

$$\longrightarrow \hat{B}^{\{e\}} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & ib \\ 0 & -ib & 0 \end{pmatrix}$$

b)hermitesch: $(A^\dagger =)\bar{A}^* = A$

$$A^\dagger = A \qquad B^\dagger = B$$

kompatibel: $[\hat{A}^{\{e\}}, \hat{B}^{\{e\}}] = 0$

Beide Matrizen sind hermitesch und kompatibel.

c) $\hat{A}^{\{e\}}:\ \lambda_1 = a \qquad \lambda_2 = -a$ (Entartet)

$$EV_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad EV_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad EV_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\lambda_1 = a \qquad \rightarrow \qquad EV = EV_1$

$\lambda_2 = -a \qquad \rightarrow \qquad EV = EV_2$ und EV_3

$\hat{B}^{\{e\}}:\ \lambda_1 = \lambda_2 = b$ (Entartet) $\qquad \lambda_3 = -b$

$\lambda_1 = b:$

$$EV_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad EV_2 = \begin{pmatrix}0\\i\\1\end{pmatrix} \qquad \text{zu } \lambda = b$$

$\lambda_3 = -b:$

$$EV_3 = \begin{pmatrix}0\\1\\-i\end{pmatrix}$$

$$|g_1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} = EV_1^{(b)} \text{ und } EV_3^{(a)}$$

$$|g_2\rangle = i \cdot \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{EV_2^{(a)}} + \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{EV_3^{(a)}} = EV_2^{(b)} \qquad \text{Linearkombination aus anderen EV}$$

$$|g_3\rangle = \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{EV_2^{(a)}} - i \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{EV_3^{(a)}}$$

$\longrightarrow$

$$|g_1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad |g_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\i\\1\end{pmatrix} \qquad |g_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\end{pmatrix} \qquad \text{mit } \langle g_i|g_j\rangle = \delta_{ij}$$

d)

$$\hat{A}^{\{e\}}\,|g_1\rangle = a\,|g_1\rangle$$

$$\hat{A}^{\{e\}}\,|g_2\rangle = \hat{A}^{\{e\}}\Big(-i\;EV_2^{(a)} + EV_3^{(a)}\Big) = \frac{1}{\sqrt{2}}\cdot\Big(i(-a)EV_2^{(a)} + (-a)EV_3^{(a)}\Big) = (-a)\,|g_2\rangle$$

$$\hat{A}^{\{e\}}\,|g_3\rangle = \hat{A}^{\{e\}}\Big(-i\;EV_3^{(a)} + EV_2^{(a)}\Big) = \frac{1}{\sqrt{2}}\cdot\Big(iaEV_3^{(a)} - aEV_2^{(a)}\Big) = (-a)\,|g_3\rangle$$

$$\hat{B}^{\{e\}}\,|g_1\rangle = b\,|g_1\rangle$$
$$\hat{B}^{\{e\}}\,|g_2\rangle = b\,|g_2\rangle$$
$$\hat{B}^{\{e\}}\,|g_3\rangle = -b\,|g_3\rangle$$

$$\longrightarrow \qquad (\hat{A}^{\{g\}})_{jk} = \langle g_j|\,\hat{A}\,|g_k\rangle \qquad (\hat{B}^{\{g\}})_{jk} = \langle g_j|\,\hat{B}\,|g_k\rangle$$

$$A^{\{g\}}\begin{pmatrix}a & 0 & 0\\0 & -a & 0\\0 & 0 & -a\end{pmatrix} \qquad B^{\{g\}} = \begin{pmatrix}b & 0 & 0\\0 & b & 0\\0 & 0 & -b\end{pmatrix}$$

In der Basis der EV sind die EW in der Diagonale der jeweiligen Operatoren

3) ∞-tiefer Potentialtopf

$$t = 0 \rightarrow WF: \quad \psi(x, t=0) = C \cdot \Big[2 \cdot sin(k_1 x) + sin(k_2 x) + 3 \cdot sin(k_3 x)\Big] \qquad k_n = \frac{n\pi}{L}$$

a)

$$E_n = \frac{\hbar^2 \pi^2}{2mL^2} \cdot n^2 \qquad \phi_n(x) = \sqrt{\frac{2}{L}}\, sin(n\, \frac{\pi}{L}\, x)$$

$$\longrightarrow \qquad \psi(x, t=0) = C \cdot \Big[2 \cdot |\phi_1\rangle + |\phi_2\rangle + 3 \cdot |\phi_3\rangle\Big]$$

$$\langle\psi|\psi\rangle = \frac{C^2 L}{2} \cdot \Big[2^2 \langle\psi_1|\psi_1\rangle + \langle\psi_2|\psi_2\rangle + 3^2 \cdot \langle\psi_3|\psi_3\rangle\Big] = \frac{C^2 L}{2} \cdot \Big(4 + 1 + 9\Big) = 7\, C^2\, L \overset{!}{=} 1$$

$$\longrightarrow C = \frac{1}{\sqrt{7L}}$$

b)

$$\psi(x, t=0) = \frac{1}{\sqrt{14}} \Big(2\, |\phi_1\rangle + |\phi_2\rangle + 3\, |\phi_3\rangle\Big)$$

$$|\psi\rangle = \mathbb{1}\, |\psi\rangle = \sum_i\, |\phi_1\rangle\, \underbrace{\langle\phi_1|\psi\rangle}_{=a_i} = \sum_i a_i\, |\phi_i\rangle$$

$$\text{Zeitentwicklung: } \psi(x,t) = \sum_n a_n\, |\phi_n\rangle \cdot e^{-i\frac{E_n}{\hbar}t}$$

$$|\phi(t)\rangle = a_1\, |\phi_1\rangle \cdot e^{-i\frac{E_1}{\hbar}t} + a_2\, |\phi_2\rangle \cdot e^{-i\frac{E_2}{\hbar}t} + a_3\, |\phi_3\rangle \cdot e^{-i\frac{E_3}{\hbar}t} + \dots$$

$$a_1 = \langle\phi_1|\psi\rangle = \frac{2}{\sqrt{14}} \qquad a_2 = \frac{1}{\sqrt{14}} \qquad a_3 = \frac{3}{\sqrt{14}}$$

$$|\psi(t)\rangle = \frac{2}{\sqrt{14}} \cdot |\phi_1\rangle \cdot e^{-i\frac{E_1}{\hbar}t} + \frac{1}{\sqrt{14}} \cdot |\phi_2\rangle \cdot e^{-i\frac{E_2}{\hbar}t} + \frac{3}{\sqrt{14}} \cdot |\phi_3\rangle \cdot e^{-i\frac{E_3}{\hbar}t}$$

$$\langle\psi(t)| = \frac{2}{\sqrt{14}} \cdot |\phi_1\rangle \cdot e^{i\frac{E_1}{\hbar}t} + \frac{1}{\sqrt{14}} \cdot |\phi_2\rangle \cdot e^{i\frac{E_2}{\hbar}t} + \frac{3}{\sqrt{14}} \cdot |\phi_3\rangle \cdot e^{i\frac{E_3}{\hbar}t}$$

$$|\psi(t)|^2 = \psi(t)^* \psi(t)$$

c)

$$\psi(x,T) = \psi(x,0) \qquad ?$$

$$\psi(x,T) = \sum_n a_n \phi_n(x) \cdot e^{-i\frac{E_n}{\hbar}t} = \sum_n a_n \phi_n(x) \cdot \underbrace{e^{-i\frac{E_n}{\hbar}0}}_{=1}$$

$$\phi_n(x) \cdot e^{-i\frac{E_n}{\hbar}t} = \phi_n(x) \qquad \longrightarrow \qquad e^{-i\frac{E_n}{\hbar}t} = e^0 = e^{\pm i2\pi}$$

$$-i\frac{E_n}{\hbar}T = i2\pi \qquad \longrightarrow \qquad T = \frac{2\pi}{E_n}\hbar$$

d)

Energiemessung: $\langle\psi|\,\hat{H}\,|\psi\rangle \qquad \longrightarrow \qquad E_1, E_2$ *u.* E_3 können gemessen werden

$$\langle\psi|\psi\rangle \stackrel{!}{=} 1 \qquad \longrightarrow \qquad |\psi\rangle = \frac{1}{\sqrt{14}}\Big(2\,|\phi_1\rangle + |\phi_2\rangle + 3\,|\phi_3\rangle\Big)$$

$$\langle\psi|\psi\rangle = \frac{1}{14} \cdot \Big[\underbrace{(a_1)^2}_{2^2}\langle\phi_1|\phi_1\rangle + \underbrace{(a_2)^2}_{1^2}\langle\phi_2|\phi_2\rangle + \underbrace{(a_3)^2}_{3^2}\langle\phi_3|\phi_3\rangle\Big]$$

$E_1 \qquad \longrightarrow \qquad$ Wahrsch.: $W(E_1) = \dfrac{4}{14} \ldots \langle\phi_1|\psi\rangle$

$E_2 \qquad \longrightarrow \qquad$ Wahrsch.: $W(E_2) = \dfrac{1}{14} \ldots \langle\phi_2|\psi\rangle$

$E_3 \qquad \longrightarrow \qquad$ Wahrsch.: $W(E_3) = \dfrac{9}{14} \ldots \langle\phi_3|\psi\rangle$

$$\sum_{i=1}^{3} W(E_i) = 1$$

e)

$$\langle E_n \rangle = \langle \psi | \hat{H} | \psi \rangle \qquad \langle \psi | \hat{H} | \psi \rangle = \sum_n E_n P_n = | \langle \phi_n | \psi \rangle |^2$$

$$\hat{H} = \sum_i |\phi_i\rangle E_i \langle \phi_i|$$

$$\sum_n E_n \cdot P_n = \frac{\pi^2 \hbar^2}{2mL^2} \cdot \left(\frac{4}{14} \underbrace{(1^2)}_{(n_1)^2} + \frac{1}{14} \underbrace{(2^2)}_{(n_2)^2} + \frac{9}{14} \underbrace{(3^2)}_{(n_3)^2} \right) = \frac{\pi^2 \hbar^2}{2mL^2} \cdot \left(\frac{89}{14} \right)$$

$$\sum_n E_n \cdot P_n \qquad \ldots \qquad \text{zeitl. konstant (“Energieerhaltung”)}$$