8. Tutorium - Quantentheorie I - 21.12.2012 - Lösung

1 a)

$$\psi(r,\theta,\phi) = r\cdot e^{-(\frac{r}{a})^2}\cdot\left(\sqrt{\frac{4\pi}{3}}Y_{1,0}+\sqrt{\frac{8\pi}{3}}\cdot\left(\frac{i-1}{2i}\right)\cdot Y_{1,-1}-\sqrt{\frac{8\pi}{3}}\cdot\left(\frac{i+1}{2i}\right)\cdot Y_{1,1}\right)$$

b)

$$\hat{\vec{L}}^2 Y_{lm}(\theta,\phi) = l(l+1)\hbar^2 Y_{lm}(\theta,\phi)$$

$$\hat{\vec{L}}_z Y_{lm}(\theta,\phi) = m\hbar Y_{lm}(\theta,\phi)$$

Normieren:

$$\langle\psi|\psi\rangle = \iint |\psi(r,\theta,\phi)|^2 r^2\ dr\ d\Omega$$

$$Y_{lm} \longrightarrow\ VONS\ : \int \bar{Y}_{l'm'}(\theta,\phi)\cdot Y_{lm}(\theta,\phi)d\Omega = \delta_{l'l}\delta_{m'm}$$

$$\langle\psi|\psi\rangle = \ldots = \frac{4\pi}{N^2}\iiint r^4\cdot e^{-2(\frac{r}{a})^2} sin(\theta)\ dr\ d\theta\ d\phi = \frac{3\sqrt{\pi^7}}{2\sqrt{2}a^5}\cdot\frac{1}{N^2} \stackrel{!}{=} 1$$

$$\psi(r,\theta,\phi) = \frac{1}{N}\cdot\left\{r\cdot e^{-(\frac{r}{a})^2}\cdot\left(\sqrt{\frac{4\pi}{3}}Y_{1,0}+\sqrt{\frac{8\pi}{3}}\cdot\left(\frac{i-1}{2i}\right)\cdot Y_{1,-1}-\sqrt{\frac{8\pi}{3}}\cdot\left(\frac{i+1}{2i}\right)\cdot Y_{1,1}\right)\right\}$$

$$\text{mit}\ \ N = \frac{2\sqrt{2}\cdot a^5}{3\sqrt{\pi^7}}$$

c) Jede Fkt. darstellbar als:

$$\psi(r,\theta,\phi) = \sum_{n;l,m} c_{nlm}\cdot R_{nl}(r)\cdot Y_{lm}(\theta,\phi)$$

$$\Longrightarrow\ a_{lm}(r) = \sum_{n} c_{nml}\cdot R_{nl}(r)$$

z.B. Wahrsch. v. $\hat{L}_z$ zu messen:

$$P(L_z = m\hbar) = \sum_{l\geq|m|}\int_0^\infty r^2\left|a_{lm}(r)\right|^2 dr$$

2 a)

$$W(r) = \int\limits_{\theta=0}^{\pi} \int\limits_{\phi=0}^{2\pi} |\psi|^2 \; r^2 \; dr \; sin(\theta) \; d\theta \; d\phi$$

$$\langle r \rangle = \int\limits_{r=0}^{\infty} \frac{r}{\pi {a_0}^3} \cdot 4\pi r^2 \; e^{-\frac{2r}{a_0}} \; dr = \ldots = \frac{3}{2} a_0$$

b)

$$\langle r^2 \rangle \;\longrightarrow\; \sqrt{\langle r^2 \rangle - {\langle r \rangle}^2}$$

$$r^2 |\psi|^2 = r^2 \cdot \frac{1}{\pi {a_0}^3} \cdot e^{-\frac{2r}{a_0}}$$

$$\frac{d}{dr} r^2 |\psi|^2 = 0 \;\longrightarrow\; r = a_0$$

c)

$$W(0 < r < a_0) = \int\limits_{r=0}^{a_0} \frac{1}{\pi {a_0}^3} \cdot 4\pi r^2 \cdot e^{-\frac{2r}{a_0}} \; dr = \ldots = \left(-\frac{5}{e^2} + 1 \right)$$