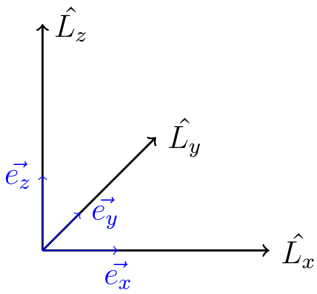
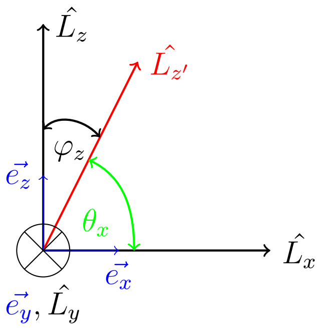
9. Tutorium - Quantentheorie I - 11.01.2013 - Lösung

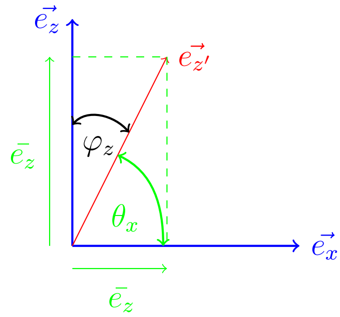
1 **allg. gilt:**

$$\vec{L} = \vec{r} \times \vec{p} = det \begin{pmatrix} \vec{e_1} & x & \hat{p_x} \\ \vec{e_2} & y & \hat{p_y} \\ \vec{e_3} & z & \hat{p_z} \end{pmatrix} = \vec{e_1} \cdot y \cdot \hat{p_z} + x \cdot \hat{p_y} \cdot \vec{e_3} + \hat{p_x} \cdot \vec{e_2} \cdot z - \vec{e_3} \cdot y \cdot \hat{p_x} - z \cdot \hat{p_y} \cdot \vec{e_1} - \hat{p_z} \cdot \vec{e_2} \cdot x =$$

$$= \underbrace{(y \cdot \hat{p_z} - z \cdot \hat{p_y})}_{\hat{L_x}} \cdot \vec{e_1} + \underbrace{(\hat{p_x} \cdot z - \hat{p_z} \cdot x)}_{\hat{L_y}} \cdot \vec{e_2} + \underbrace{(x \cdot \hat{p_y} - y \cdot \hat{p_x})}_{\hat{L_z}} \cdot \vec{e_3}$$

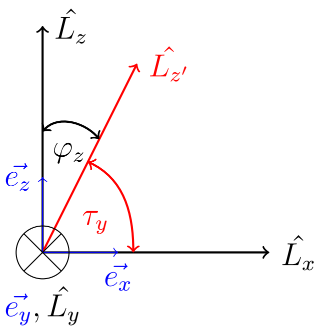


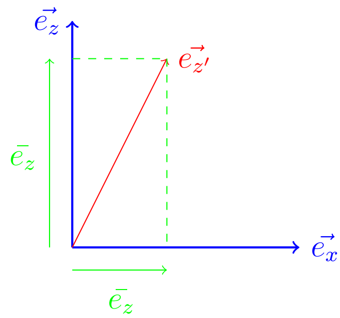




blau . . . kartesische Basisvektoren
gruen . . . Basisvektoren bezogen auf die Achse z'

$$cos(\theta_x) = \frac{\bar{e_x}}{\vec{e_x}} \qquad cos(\varphi_z) = \frac{\bar{e_z}}{\vec{e_z}} \qquad \Longrightarrow \vec{e_{z'}} = \bar{e_x} + \bar{e_z} = cos(\theta_x) \cdot \vec{e_x} + cos(\varphi_z) \cdot \vec{e_z}$$





$$cos(\tau_y) = \frac{\bar{e_y}}{\vec{e_y}} \qquad \Longrightarrow \qquad \bar{e_y} = cos(\tau_y) \cdot \vec{e_y}$$

Der neue Vektor als Funktion der alten kartesischen Basisvektoren:

$$\vec{e_{z'}} = \bar{e_x} + \bar{e_y} + \bar{e_z} = cos(\theta_x) \cdot \vec{e_x} + cos(\varphi_z) \cdot \vec{e_z} + cos(\tau_y) \cdot \vec{e_y}$$

$$\vec{e_{z'}} = \begin{pmatrix} cos(\theta_x) \cdot \vec{e_x} \\ cos(\varphi_z) \cdot \vec{e_z} \\ cos(\tau_y) \cdot \vec{e_y} \end{pmatrix} \qquad \Longrightarrow \vec{L_{z'}} = det \begin{pmatrix} cos(\theta_x) \cdot \vec{e_x} & x & \hat{p_x} \\ cos(\varphi_z) \cdot \vec{e_z} & y & \hat{p_y} \\ cos(\tau_y) \cdot \vec{e_y} & z & \hat{p_z} \end{pmatrix} =$$

$$= cos(\theta_x) \cdot \vec{e_x} \cdot y \cdot \hat{p_z} + x \cdot \hat{p_y} \cdot cos(\varphi_z) \cdot \vec{e_z} + \hat{p_x} \cdot cos(\tau_y) \cdot \vec{e_y} \cdot z -$$

$$-cos(\varphi_z) \cdot \vec{e_z} \cdot y \cdot \hat{p_x} - z \cdot \hat{p_y} \cdot cos(\theta_x) \cdot \vec{e_x} - \hat{p_z} \cdot cos(\tau_y) \cdot \vec{e_y} \cdot x =$$

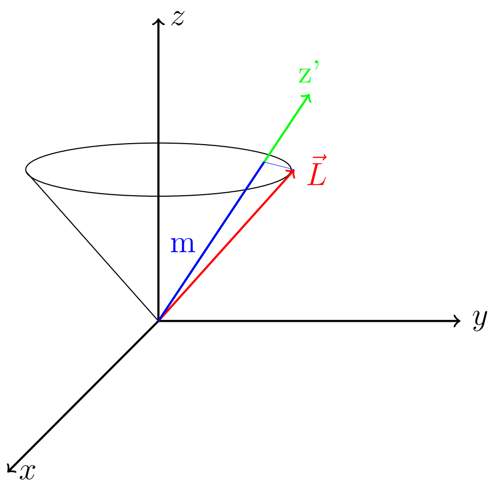
$$= \Big(cos(\theta_x) \cdot y \cdot \hat{p_z} - z \cdot \hat{p_y} \cdot cos(\theta_x)\Big) \cdot \vec{e_x} + \Big(\hat{p_x} \cdot cos(\tau_y) \cdot z - \hat{p_z} \cdot cos(\tau) \cdot x\Big) \cdot \vec{e_y} +$$

$$+\Big(x \cdot \hat{p_y} \cdot cos(\varphi_z) - cos(\varphi_z) \cdot y \cdot \hat{p_x}\Big) \cdot \vec{e_z} =$$

$$= \Big[cos(\theta_x) \cdot \underbrace{\Big(y \cdot \hat{p_z} - z \cdot \hat{p_y}\Big)}_{\hat{L_x}}\Big] \cdot \vec{e_x} + \Big[cos(\tau_y) \cdot \underbrace{\Big(\hat{p_x} \cdot z - \hat{p_z} \cdot x\Big)}_{\hat{L_y}}\Big] \cdot \vec{e_y} + \Big[cos(\varphi_z) \cdot \underbrace{\Big(x \cdot \hat{p_y} - y \cdot \hat{p_x}\Big)}_{\hat{L_z}}\Big] \cdot \vec{e_z}$$

$$\Longrightarrow \quad \hat{L_{z'}} = cos(\theta_x) \cdot \hat{L_x} \cdot \vec{e_x} + cos(\tau_y) \cdot \hat{L_y} \cdot \vec{e_y} + cos(\varphi_z) \cdot \hat{L_z} \cdot \vec{e_z}$$

$$\langle\psi| \hat{L_{z'}} |\psi\rangle = cos(\theta_x) \cdot \langle\psi| \hat{L_x} |\psi\rangle + cos(\tau_y) \cdot \langle\psi| \hat{L_y} |\psi\rangle + cos(\varphi_z) \cdot \langle\psi| \hat{L_z} |\psi\rangle = cos(\varphi_z) \cdot \hbar m$$



2 a) **angeregter Zustand:** $n = 2$

$$E_n = -\frac{\mu z^2 e^4}{8\varepsilon_0 h^2 n^2} \qquad z = 1;\ \mu \ldots \text{reduzierte Masse}$$

$$\Delta E_n = E_2 - E_1 = \frac{\mu e^4}{8\varepsilon_0 h^2}\left(\frac{1}{{n_1}^2} - \frac{1}{{n_2}^2}\right) = 1.634 \cdot 10^{-18}\ J$$

$$\Delta E = hf = \hbar\omega$$

$$f = \frac{3\mu e^4}{32 e_0^2 h^2} = 2.466 \cdot 10^{15}\ Hz$$

$$c = \lambda f \longrightarrow \lambda = \frac{c}{f} = 1.215 \cdot 10^{-7} m = 121.57\ nm(UV)$$

b) Eigenzustände $|nlm\rangle$ des Wasserstoffatoms $\longrightarrow$ el. Dipolmoment ?
$\longrightarrow$ Erwartungswert $\langle nlm|\,\vec{d}\,|nlm\rangle$ des Dipoloperators $\vec{d} = -\vec{e_r}$

$$\langle nlm|\,\vec{d}\,|nlm\rangle = \langle nlm|\,\underbrace{\pi\pi^{-1}}_{=\mathbb{1}}|nlm\rangle = \langle nlm|\,(-1)^l\,\underbrace{\pi\vec{d}\pi^{-1}}_{-\vec{d}}|nlm\rangle =$$

$$= \langle nlm|\,\underbrace{(-1)^{2l}}_{=1\ \ldots\ l\in\mathbb{N}}(-1)\vec{d}\,|nlm\rangle = -(-1)^{2l}\,\langle nlm|\,\vec{d}\,|nlm\rangle = -\,\langle nlm|\,\vec{d}\,|nlm\rangle$$

$$\pi\,|nlm\rangle = (-1)^l\,|nlm\rangle$$

$$\longrightarrow \langle nlm|\,\vec{d}\,|nlm\rangle = -\,\langle nlm|\,\vec{d}\,|nlm\rangle = 0$$

3 a) Spin $s = \frac{1}{2} \longrightarrow$ Zustand: $\chi = \frac{\sqrt{3}}{s} \left|+\right\rangle + \frac{i}{2} \left|-\right\rangle$

$\left|+\right\rangle, \left|-\right\rangle$ ... EZ v. $\hat{S}_z$ zu EW $\pm\frac{\hbar}{2}$

$$\hat{S}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad EW : \begin{cases} -\frac{\hbar}{2} \longrightarrow EV : \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left|-\right\rangle & \left\langle -\right| = \begin{pmatrix} 0 & 1 \end{pmatrix} \\ \frac{\hbar}{2} \longrightarrow EV : \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left|+\right\rangle & \left\langle +\right| = \begin{pmatrix} 1 & 0 \end{pmatrix} \end{cases}$$

Wahrsch. bei Messung des Spins in z-Richtung $-\frac{\hbar}{2}$ zu erhalten?

$$\left| \left\langle \chi | - \right\rangle \right|^2 = \left| \left(\frac{\sqrt{3}}{2} \left\langle +\right| - \frac{i}{2} \left\langle -\right| \right) \cdot \left(\left|-\right\rangle \right) \right|^2 = \frac{1}{4}$$

b)

$$\hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar i}{2} \left(\left|-\right\rangle \left\langle +\right| - \left|+\right\rangle \left\langle -\right| \right)$$

$$\langle \hat{S}_y \rangle = \langle \chi | \hat{S}_y | \chi \rangle = \left(\left\langle +\right| \frac{\sqrt{3}}{2} - \frac{i}{2} \left\langle -\right| \right) \cdot \left(\frac{\hbar i}{2} \left(\left|-\right\rangle \left\langle +\right| - \left|+\right\rangle \left\langle -\right| \right) \cdot \left(\frac{\sqrt{3}}{2} \left|+\right\rangle + \frac{i}{2} \left|-\right\rangle \right) \right) =$$

$$= \ldots = -\frac{i\hbar\sqrt{3}}{4} \cdot (i) \longrightarrow \langle \hat{S}_y \rangle^2 = \frac{3\hbar^2}{16}$$

$$\langle \hat{S}_y^2 \rangle = \langle \chi | \hat{S}_y^2 | \chi \rangle = \left(\frac{\sqrt{3}}{2} \left\langle +\right| - \frac{i}{2} \left\langle -\right| \right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \left(\frac{\sqrt{3}}{2} \left|+\right\rangle + \frac{i}{2} \left|-\right\rangle \right) = \ldots = \frac{\hbar^2}{4}$$

$$\Delta \hat{S}_y = \sqrt{\langle \hat{S}_y^2 \rangle - \langle \hat{S}_y \rangle^2} = \ldots = \frac{\hbar}{4}$$

c) **Blochkugel:**

$$\left|\psi\right\rangle = cos\left(\frac{\theta}{2}\right) \left|0\right\rangle + e^{i\varphi} sin\left(\frac{\theta}{2}\right) \left|1\right\rangle$$

$$\frac{\hbar}{2} \longrightarrow \text{messen} \longrightarrow \text{EV} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left|0\right\rangle$$

$$\varphi = \frac{\sqrt{3}}{2} \left|+\right\rangle + \frac{i}{2} \left|-\right\rangle$$

$$cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2} \longrightarrow \frac{\theta}{2} = \frac{\pi}{6} \longrightarrow \theta = \frac{\pi}{3} = 60°$$

$$e^{i\varphi} \cdot sin\left(\frac{\theta}{2}\right) = \frac{i}{2} = i \cdot \frac{1}{2}$$

$$sin\left(\frac{\theta}{2}\right) = \frac{1}{2} \longrightarrow \frac{\theta}{2} = \frac{\pi}{6} \longrightarrow \theta = 60°$$

$$cos(\varphi) + isin(\varphi) = e^{\varphi i} \stackrel{!}{=} i \longrightarrow \varphi = \frac{1}{2}\left(4\pi n + \pi\right) \quad n \in \mathbb{Z}$$

$$\varphi = \frac{\pi}{2} = 90°$$

4 a) **Eigenwerte:**

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \longrightarrow \text{EW:} \quad \lambda = \pm 1 \qquad E_1 = \pm\frac{\hbar}{2}$$

Eigenvektoren:

$\lambda_1 = \frac{\hbar}{2}$

$$\begin{pmatrix} -\frac{\hbar}{2} & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & -\frac{\hbar}{2} \end{pmatrix} \vec{x} = \vec{0} \longrightarrow EV_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$\lambda_2 = -\frac{\hbar}{2}$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \vec{x} = \vec{0} \longrightarrow EV_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\hat{S}_y \left|\chi(0)\right\rangle = \frac{\hbar}{2}\left|\chi(0)\right\rangle$$

$$\hat{S}_y \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

b)

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow \hat{S}_x \left|\pm\right\rangle_x = \pm\frac{\hbar}{2}\left|\pm\right\rangle_x \longrightarrow \text{EV v. } \hat{S}_x \text{ für } \pm\frac{\hbar}{2} : \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

$$\left|\pm\right\rangle_x = \frac{1}{\sqrt{2}}\Big(\left|+\right\rangle \pm \left|-\right\rangle\Big)$$

Zeitentwicklung von $\left|\chi(0)\right\rangle$

$$W\Big(+\frac{\hbar}{2}\Big) = \Big|\left\langle + \right|_x \chi(t)\rangle\Big|^2 \stackrel{!}{=} 1$$

$$\Big|\frac{1}{\sqrt{2}}\Big(\left\langle + \right| + \left\langle - \right|\Big)\cdot\Big(e^{-\frac{i\gamma Bt}{2}}\left|+\right\rangle + i\cdot e^{\frac{i\gamma Bt}{2}}\left|-\right\rangle\Big)\Big|^2 = \ldots = \frac{1}{2} - \frac{1}{2}sin(\gamma Bt) \stackrel{!}{=} 1$$

$$\text{wobei} \qquad sin(\varphi) = \frac{e^{i\varphi} - e^{-i\varphi}}{2} \text{ benützt wurde}$$

$$1 - sin(\gamma Bt) = 2 \longrightarrow sin(\gamma Bt) = -1$$

und somit erhalten wir:

$$B = \frac{3\pi}{2\gamma t}$$