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a) $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 + \alpha \hat{x}^2 = \underline{\underline{\frac{\hat{p}^2}{2m} + \frac{1}{2} m\tilde{\omega}^2 x^2}}$ $\quad \tilde{\omega}^2 = \left(\omega^2 + \frac{2\alpha}{m}\right)$

$\underline{\underline{\tilde{E}_n = \hbar\tilde{\omega}(n + 1/2)}}$

b) $E_n^{(1)} = \langle \Psi_n^{(0)} | V | \Psi_n^{(0)} \rangle$

$V = \alpha \hat{x}^2 \qquad \hat{x} = \frac{x_0}{\sqrt{2}}(\hat{a} + \hat{a}^+) \qquad x_0 = \sqrt{\frac{\hbar}{m\omega}}$

$V = \frac{x_0^2}{2}(2\hat{n} + 1 + \hat{a}\hat{a} + \hat{a}^+\hat{a}^+)$

$E_n^{(1)} = \frac{\alpha\hbar}{2m\omega} \langle n^{(0)} | 2\hat{n} + 1 + \hat{a}\hat{a} + \hat{a}^+\hat{a}^+ | n^{(0)} \rangle = \underline{\underline{\frac{\alpha\hbar}{m\omega}(n + 1/2)}}$

$$E_n^{(2)} = \sum_{\epsilon_m \neq \epsilon_n} \frac{|\langle n^{(0)} | 2\hat{n} + 1 + \hat{a}\hat{a} + \hat{a}^+\hat{a}^+ | m \rangle|^2}{\epsilon_n - \epsilon_m} \cdot \frac{\alpha^2\hbar^2}{4m^2\omega^2}$$

$\epsilon_m \neq \epsilon_n \Rightarrow \quad m = n \pm 2$

$$E_n^{(2)} = \frac{\alpha^2\hbar^2}{4m^2\omega^2} \left\{ \frac{|\langle n^{(0)} | \hat{a}\hat{a} | n+2^{(0)} \rangle|^2}{\hbar\omega(n+\frac{1}{2}) - \hbar\omega(n+2+\frac{1}{2})} + \frac{|\langle n^{(0)} | \hat{a}\hat{a} | n-2^{(0)} \rangle|^2}{\hbar\omega(n+\frac{1}{2}) - \hbar\omega(n-2+\frac{1}{2})} \right\} =$$

$$= \underline{\underline{-\frac{\alpha^2\hbar}{2m^2\omega^3}(n + 1/2)}}$$

c) $$|n^{(1)}\rangle = \sum_{m \neq n} |m^{(0)}\rangle \frac{\langle m^{(0)} | V | n^{(0)} \rangle}{\mathcal{E}_n - \mathcal{E}_m} \qquad m = n \pm 2$$

$$= \frac{\alpha\hbar}{2m\omega} \left\{ \frac{\langle n+2^{(0)} | \hat{a}^+\hat{a}^+ | n^{(0)} \rangle}{-2\hbar\omega} | n+2^{(0)} \rangle \right.$$

$$\left. + \frac{\langle n-2^{(0)} | \hat{a}\hat{a} | n^{(0)} \rangle}{2\hbar\omega} | n-2^{(0)} \rangle \right\} =$$

$$= \underline{\underline{\frac{\alpha}{4m\omega^2} \left\{ \sqrt{n-1}\sqrt{n}\, | n-2^{(0)} \rangle - \sqrt{n+1}\sqrt{n+2}\, | n+2^{(0)} \rangle \right\}}}$$

d) $E_{n,\text{gestört}} = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \dots$

$$= \hbar\omega(n+1/2) + \frac{\alpha\hbar}{m\omega}(n+1/2) - \frac{\alpha^2\hbar}{2m^2\omega^3}(n+1/2) =$$

$$= \underline{\underline{\hbar\omega(n+1/2)\left[1 + \frac{\alpha}{m\omega^2} - \frac{\alpha^2}{2m^2\omega^4}\right]}}$$

Vergleich exakte Lsg.

$$\tilde{E}_n = \hbar\omega(n+1/2)\sqrt{1+\frac{2\alpha}{m\omega^2}} \approx \underline{\underline{\hbar\omega(n+\tfrac{1}{2})\left[1+\frac{\alpha}{m\omega^2} - \frac{\alpha^2}{2m^2\omega^4}\right.}} + O\left(\left(\frac{2\alpha}{m\omega^2}\right)^3\right)\Big]$$

Taylorentwicklung der Wurzel in $\frac{2\alpha}{m\omega^2}$

D.h. passt für kleine α

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a) $|\tilde{n}_e^{(0)}\rangle = |e\rangle|n\rangle \qquad E_{\tilde{n},e}^{(0)} = \varepsilon + \hbar\omega(n+\tfrac{1}{2})$

$|\tilde{n}_g^{(0)}\rangle = |g\rangle|n\rangle \qquad E_{\tilde{n},g}^{(0)} = \hbar\omega(n+\tfrac{1}{2})$

b) $\langle n|m\rangle = \delta_{mn} \qquad \langle g|e\rangle = \delta_{ge}$

$$E_{\tilde{n},e}^{(1)} = \langle \tilde{n}_e^{(0)}|\hat{H}_I|\tilde{n}_e^{(0)}\rangle = \alpha\Big[\underbrace{\langle n|\hat{a}+\hat{a}^+|n\rangle}_{=0}\ \underbrace{\langle e|(|e\rangle\langle g| + |g\rangle\langle e|)|e\rangle}_{=0}\Big] =$$

$$= \underline{\underline{0}}$$

$E_{\tilde{n},g}^{(1)} = \underline{\underline{0}}$ s. Zeile darüber

$$E_{\tilde{n},e}^{(2)} = \sum_{\varepsilon_{\tilde{n}} \neq \varepsilon_{\tilde{m}}} \frac{|\langle \tilde{n}_e^{(0)}|H_I|\tilde{m}^{(0)}\rangle|^2}{\varepsilon_{\tilde{n}e} - \varepsilon_{\tilde{m}}}$$

$|\tilde{n}_e^{(0)}\rangle = |e\rangle|n\rangle$

nur Beiträge für $|\tilde{m}\rangle = |g\rangle|n+1\rangle$

od $|\tilde{m}\rangle = |g\rangle|n-1\rangle$

$$E^{(2)}_{\tilde{n},e} = d^2 \left[\frac{|\langle n|\hat{a}^+|n-1\rangle|^2}{\varepsilon + \hbar\omega(n+\frac{1}{2}) - (\hbar\omega(n-1+\frac{1}{2}))} + \frac{|\langle n|\,\hat{a}\,|n+1\rangle|^2}{\varepsilon + \hbar\omega(n+\frac{1}{2}) - \hbar\omega(n+1+\frac{1}{2})} \right] =$$

$$= d^2 \left[\frac{n}{\varepsilon + \hbar\omega} + \frac{n+1}{\varepsilon - \hbar\omega} \right] = \underline{\underline{-d^2 \frac{\hbar\omega + \varepsilon(2n+1)}{\hbar^2\omega^2 - \varepsilon^2}}}$$

$E^{(2)}_{\tilde{n},g}$ nur Beiträge für $|\tilde{m}\rangle = |e\rangle|n+1\rangle$

od $|\tilde{m}\rangle = |e\rangle|n-1\rangle$

$$E^{(2)}_{\tilde{n},g} = d^2 \left[\frac{n}{\hbar\omega - \varepsilon} + \frac{n+1}{-\hbar\omega - \varepsilon} \right] = \underline{\underline{-d^2 \frac{\hbar\omega - \varepsilon(2n+1)}{\hbar^2\omega^2 - \varepsilon^2}}}$$

c) $B = \{ |g,1\rangle, |e,0\rangle \}$ $\quad \varepsilon = \hbar\omega$

$$H_I = d(|e\rangle\langle g| + |g\rangle\langle e|)(\hat{a} + \hat{a}^+)$$

in Basis B

$$\langle g,1|H_I|g,1\rangle = \langle e,0|H_I|e,0\rangle = 0$$

$$\langle g,1|H_I|e,0\rangle = d\langle g,1|(|g\rangle\langle e|a^+)|e,0\rangle = \underline{\underline{d}} = \langle e,0|H_I|g,1\rangle$$

$$\underline{\underline{H_I^{\{B\}} \mathrel{\hat{=}} \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}_{\{B\}}}}$$

Eigenwerte $\left\{ \begin{array}{l} \underline{\lambda_1 = +d} \\ \underline{\underline{\lambda_2 = -d}} \end{array} \right.$ $\quad$ Eigenvektoren $\left\{ \begin{array}{l} \underline{|\lambda_1\rangle = \frac{1}{\sqrt{2}}(|g,1\rangle + |e,0\rangle)} \\ \underline{\underline{|\lambda_2\rangle = \frac{1}{\sqrt{2}}(|g,1\rangle - |e,0\rangle)}} \end{array} \right.$

$\Rightarrow$ Aufspaltung

$|g,1\rangle$, $|e,0\rangle$: $E = \frac{3\hbar\omega}{2}$

$\frac{1}{\sqrt{2}}(|g,1\rangle + |e,0\rangle)$: $E = \frac{3\hbar\omega}{2} + d$

$\frac{1}{\sqrt{2}}(|g,1\rangle - |e,0\rangle)$: $E = \frac{3\hbar\omega}{2} - d$