5. Tutorium VU Quantentheorie I – Lösungen, 20.11.2015

1.
   - Nach Verwendung der Definition von Kommutator bzw. Anti-Kommutator und Vereinfachen der rechten Seite der Gleichungen sieht man die Gültigkeit der angegebenen Beziehungen für die Operatoren.
   - $\left[\hat{x}, \hat{H}\right] = \frac{i\hbar}{m}\hat{p} \Rightarrow \langle\phi|\hat{p}|\phi\rangle = \langle\phi|\frac{m}{i\hbar}\left[\hat{x}, \hat{H}\right]|\phi\rangle = 0$

2. a) $\langle a_i \mid a_j\rangle = \delta_{ij} \Rightarrow \langle b_i \mid b_j\rangle = \delta_{ij}, \quad \mathbb{1} = \sum_{i=1}^{3} |a_i\rangle\langle a_i| = \sum_{i=1}^{3} |b_i\rangle\langle b_i|$

   b) $|a_1\rangle \overset{\{a\}}{\to} a_1^{\{a\}}=\binom{1}{0}, \quad |a_2\rangle \overset{\{a\}}{\to} a_2^{\{a\}}=\binom{0}{1}, \quad |b_1\rangle \overset{\{a\}}{\to} b_1^{\{a\}}=\frac{1}{\sqrt{2}}\binom{1}{i}, \quad |b_2\rangle \overset{\{a\}}{\to} b_2^{\{a\}}=\frac{1}{\sqrt{2}}\binom{1}{-i}$

   $\langle b_1| \overset{\{a\}}{\to} b_1^{\{a\}\dagger}=\frac{1}{\sqrt{2}}(1,-i), \quad \langle b_2| \overset{\{a\}}{\to} b_2^{\{a\}\dagger}=\frac{1}{\sqrt{2}}(1,i)$

   c) $\hat{U} = \sum_{i=1}^{2} |b_i\rangle\langle a_i|, \quad U_{mn}^{\{a\}} = \left\langle a_m \middle| \hat{U} \middle| a_n\right\rangle = \langle a_m \mid b_n\rangle, \quad \hat{U} \overset{\{a\}}{\to} U^{\{a\}} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$

   $\hat{U}\hat{U}^\dagger = \mathbb{1}$ (Normerhaltung, Invarianz von Erwartungswerten)

   d) $|\chi\rangle \overset{\{b\}}{\to} \chi^{\{b\}}=\frac{1}{2}\begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$

   $\hat{T} \overset{\{b\}}{\to} T^{\{b\}}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

   $\left\langle \chi \middle| \hat{T} \middle| \chi \right\rangle = \chi^{\{a\}\dagger}T^{\{a\}}\chi^{\{a\}} = \chi^{\{b\}\dagger}T^{\{b\}}\chi^{\{b\}} = 0$

3. a) $H_{ij}^{\{\phi\}} = \langle\phi_i|\hat{H}|\phi_j\rangle \to H^{\{\phi\}} = \begin{pmatrix} -a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 2a \end{pmatrix}$

   $B_{ij}^{\{\phi\}} = \langle\phi_i|\hat{B}|\phi_j\rangle \to B^{\{\phi\}} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & 2ib \\ 0 & -2ib & 0 \end{pmatrix}$

   $\hat{H} = \hat{H}^\dagger$, $\hat{B} = \hat{B}^\dagger$, $[\hat{H}, \hat{B}] \neq 0$

b) $\hat{H}$ ist offensichtlich bereits in seiner Eigenbasis ($\{\phi\}$)gegeben.

$$\phi_1^{\{\phi\}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad E_1 = -a$$

$$\phi_2^{\{\phi\}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad E_2 = a$$

$$\phi_3^{\{\phi\}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad E_3 = 2a$$

$$\langle \phi_i \,|\, \phi_j \rangle = \delta_{ij}$$

Lösung des Eigenwertproblems für $\hat{B}$:

$$b_1^{\{\phi\}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}, \quad B_1 = -2b$$

$$b_2^{\{\phi\}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad B_2 = b$$

$$b_3^{\{\phi\}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \quad B_3 = 2b$$

$$\langle b_i \,|\, b_j \rangle = \delta_{ij}$$

c) $W_{E_1} = |\langle \phi_1 | \chi \rangle|^2 = \frac{25}{169}$, $W_{E_2} = |\langle \phi_2 | \chi \rangle|^2 = 0$, $W_{E_3} = |\langle \phi_3 | \chi \rangle|^2 = \frac{144}{169}$, $\left\langle \chi \middle| \hat{H} \middle| \chi \right\rangle = \frac{263}{169} a$

$W_{B_1} = |\langle b_1 | \chi \rangle|^2 = \frac{72}{169}$, $W_{B_2} = |\langle b_2 | \chi \rangle|^2 = \frac{25}{169}$, $W_{B_3} = |\langle b_3 | \chi \rangle|^2 = \frac{72}{169}$, $\left\langle \chi \middle| \hat{B} \middle| \chi \right\rangle = \frac{25}{169} b$