10. Übung

26) $\vec{J} = \vec{L} + \vec{S}$

$s = \frac{1}{2} \quad m_s = \pm \frac{1}{2}$

$l = 1 \quad m_l = -1, 0, 1$

a) Eigenwerte: $m = m_l + m_s \quad |l-s| \leq j \leq l+s$

$j = \frac{3}{2} \rightarrow m_j = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

$j = \frac{1}{2} \rightarrow m_j = -\frac{1}{2}, \frac{1}{2}$

Eigenwerte von J^2: $J^2 |j, m_j\rangle = \hbar^2 j(j+1) |j, m_j\rangle$

J_z: $J_z |j, m_j\rangle = \hbar m_j |j, m_j\rangle$

b) $|j^{max}, m_j^{max}\rangle = |\frac{3}{2}, \frac{3}{2}\rangle_{jm} = |m_l = l, m_s = s\rangle = |1, \frac{1}{2}\rangle_{ls}$

andere Zustände zu $j = \frac{3}{2}$ durch Anwenden von

$J_- = L_- + S_-$: $J_- |j\, m_j\rangle = \hbar \sqrt{j(j+1) - m_j(m_j-1)}\, |j\, m_j - 1\rangle$

$J_- |\frac{3}{2}, \frac{3}{2}\rangle_{jm} = \hbar\sqrt{3}\, |\frac{3}{2}, \frac{1}{2}\rangle_{jm} = (L_- + S_-)|1, \frac{1}{2}\rangle_{ls} =$

$= \hbar\sqrt{2}\, |0, \frac{1}{2}\rangle_{ls} + \hbar\, |1, -\frac{1}{2}\rangle_{ls}$

$\Rightarrow |\frac{3}{2}, \frac{1}{2}\rangle_{jm} = \sqrt{\frac{2}{3}}\, |0, \frac{1}{2}\rangle_{ls} + \frac{1}{\sqrt{3}}\, |1, -\frac{1}{2}\rangle_{ls}$

$J_- |\frac{3}{2}, \frac{1}{2}\rangle_{jm} = 2\hbar\, |\frac{3}{2}, -\frac{1}{2}\rangle_{jm} =$

$= (L_- + S_-)\left(\sqrt{\frac{2}{3}}\, |0, \frac{1}{2}\rangle_{ls} + \frac{1}{\sqrt{3}} |1, -\frac{1}{2}\rangle_{ls}\right)$

$= \sqrt{\frac{2}{3}}\left(\hbar\sqrt{2}\, |-1, \frac{1}{2}\rangle_{ls} + \hbar\, |0, -\frac{1}{2}\rangle_{ls}\right) + \frac{1}{\sqrt{3}}\left(\hbar\sqrt{2}\, |0, -\frac{1}{2}\rangle_{ls} + 0\right)$

$= \hbar \frac{2}{\sqrt{3}} |-1, \frac{1}{2}\rangle_{ls} + 2\sqrt{\frac{2}{3}}\, \hbar\, |0, -\frac{1}{2}\rangle_{ls}$

$\Rightarrow |\frac{3}{2}, -\frac{1}{2}\rangle_{jm} = \frac{1}{\sqrt{3}} |-1, \frac{1}{2}\rangle_{ls} + \sqrt{\frac{2}{3}}\, |0, -\frac{1}{2}\rangle_{ls}$

$|j^{max}, m_j^{min}\rangle = |\frac{3}{2}, -\frac{3}{2}\rangle_{jm} = |-1, -\frac{1}{2}\rangle_{ls}$

Zustände zu $j=\frac{1}{2}$ durch Orthogonalitätsrelation:

$${}_{jm}\langle \tfrac{1}{2}, \tfrac{1}{2} | \tfrac{3}{2}, \tfrac{1}{2} \rangle_{jm} \overset{!}{=} 0$$

$$\left(\alpha\, {}_{ls}\langle 0, \tfrac{1}{2}| + \beta\, {}_{ls}\langle 1, -\tfrac{1}{2}|\right)\left(\sqrt{\tfrac{2}{3}}\,|0, \tfrac{1}{2}\rangle_{ls} + \tfrac{1}{\sqrt{3}}\,|1, -\tfrac{1}{2}\rangle_{ls}\right) = 0$$

$\alpha \sqrt{\frac{2}{3}} + \beta \frac{1}{\sqrt{3}} = 0$ und $|\alpha|^2 + |\beta|^2 = 1$ (Normierung)

$\Rightarrow \alpha = -\frac{1}{\sqrt{3}} \quad \beta = \sqrt{\frac{2}{3}}$

$$\Rightarrow |\tfrac{1}{2}, \tfrac{1}{2}\rangle_{jm} = -\tfrac{1}{\sqrt{3}}\,|0, \tfrac{1}{2}\rangle_{ls} + \sqrt{\tfrac{2}{3}}\,|1, -\tfrac{1}{2}\rangle_{ls}$$

$$J_-|\tfrac{1}{2}, \tfrac{1}{2}\rangle_{jm} = \hbar\,|\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{jm}$$
$$= (L_- + S_-)\left(-\tfrac{1}{\sqrt{3}}\,|0, \tfrac{1}{2}\rangle_{ls} + \sqrt{\tfrac{2}{3}}\,|1, -\tfrac{1}{2}\rangle_{ls}\right)$$
$$= -\tfrac{1}{\sqrt{3}}\left(\hbar\sqrt{2}\,|-1, \tfrac{1}{2}\rangle_{ls} + \hbar\,|0, -\tfrac{1}{2}\rangle_{ls}\right) + \sqrt{\tfrac{2}{3}}\,\hbar\sqrt{2}\,|0, -\tfrac{1}{2}\rangle_{ls}$$
$$= -\hbar\sqrt{\tfrac{2}{3}}\,|-1, \tfrac{1}{2}\rangle_{ls} + \hbar\tfrac{1}{\sqrt{3}}\,|0, -\tfrac{1}{2}\rangle_{ls}$$

$$\Rightarrow |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{jm} = -\sqrt{\tfrac{2}{3}}\,|-1, \tfrac{1}{2}\rangle_{ls} + \tfrac{1}{\sqrt{3}}\,|0, -\tfrac{1}{2}\rangle_{ls}$$

c) $\vec{L}\cdot\vec{S} = \frac{1}{2}(\vec{L}+\vec{S})^2 - \frac{1}{2}\vec{L}^2 - \frac{1}{2}\vec{S}^2$

$H = \xi(r)\,\vec{L}\cdot\vec{S} = \frac{1}{2}\xi(r)\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$

$$\langle j\, m_j | H | j\, m_j\rangle = \tfrac{\hbar^2}{2}\xi(r)\left(j(j+1) - l(l+1) - s(s+1)\right)$$

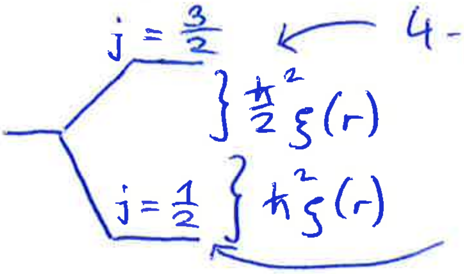
$$\langle \tfrac{3}{2}\, m_j | H | \tfrac{3}{2}\, m_j\rangle = \tfrac{\hbar^2}{2}\xi(r)$$

$$\langle \tfrac{1}{2}\, m_j | H | \tfrac{1}{2}\, m_j\rangle = -\hbar^2\xi(r)$$

d) ohne Spin-Bahn-Korrektur: 6-fache Entartung

der Zustände $\underbrace{j=\tfrac{3}{2}}_{4}$ und $\underbrace{j=\tfrac{1}{2}}_{2}$

mit Spin-Bahn-Korrektur: Aufspaltung



4-fach entartet $m_j = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

2-fach entartet $m_j = -\frac{1}{2}, \frac{1}{2}$