IV. Plenum (14.11.22)

Test von 2019 und Wdh. von Beispielen

Lösung für Übungsbeispiel 8)c)

$$(O_1 f)(x) = -i f'(x) \quad \text{und} \quad (O_2 f)(x) = x f(x)$$

$$[O_1, O_2] f(x) = \left(-i \frac{\partial}{\partial x}\right)(x f)(x) - x \left(-i \frac{\partial f}{\partial x}\right)(x)$$

$$= -i f(x) - \cancel{i x \frac{\partial f}{\partial x}(x)} + \cancel{i x \frac{\partial f}{\partial x}(x)}$$

$$= -i f(x) \Rightarrow [O_1, O_2] = -i$$

$$[O_1^2, O_2] = O_1 [O_1, O_2] + [O_1, O_2] O_1$$

$$= -2i O_1$$

$$[O_1, O_2^2] = O_2 [O_1, O_2] + [O_1, O_2] O_2$$

$$= -2i O_2$$

1. Test von 2019

1)a) $|\psi\rangle = c_0|E_0\rangle + c_1|E_1\rangle$, $1 \overset{!}{=} \langle\psi|\psi\rangle$

$$\langle\psi|\psi\rangle = \left(c_0^*\langle E_0| + c_1^*\langle E_1|\right)\left(c_0|E_0\rangle + c_1|E_1\rangle\right)$$

$$= |c_0|^2\langle E_0|E_0\rangle + |c_1|^2\langle E_1|E_1\rangle = |c_0|^2 + |c_1|^2$$

1)b) $|\psi(t_0)\rangle = c_0|E_0\rangle + c_1|E_1\rangle \qquad t_0 = 0$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}(t-t_0)H}|\psi(t_0)\rangle$$

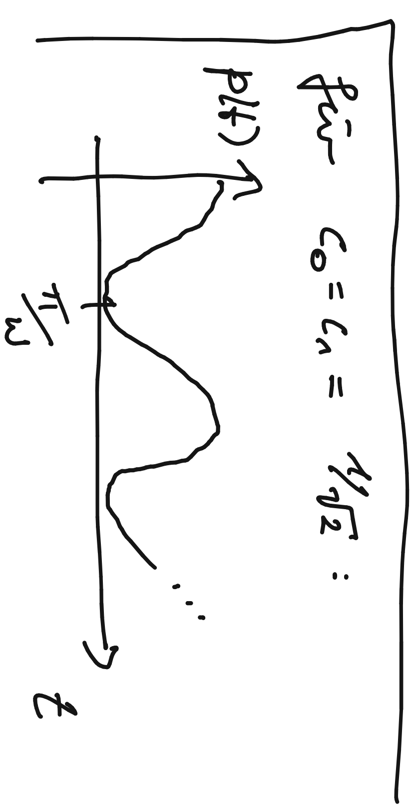
$$= c_0 e^{-\frac{i}{\hbar}tE_0}|E_0\rangle + c_1 e^{-\frac{i}{\hbar}tE_1}|E_1\rangle$$

$$= c_0 e^{+i\frac{\omega}{2}t}|E_0\rangle + c_1 e^{-i\frac{\omega}{2}t}|E_1\rangle$$

1)c) $a(t) = \langle\psi(t_0)|\psi(t)\rangle$, $p(t) = |a(t)|^2$

$$= |c_0|^2 e^{+i\frac{\omega}{2}t} + |c_1|^2 e^{-i\frac{\omega}{2}t}$$

$$p(t) = \left(|c_0|^2 e^{+i\frac{\omega}{2}t} + |c_1|^2 e^{-i\frac{\omega}{2}t}\right) \times \left(|c_0|^2 e^{-i\frac{\omega}{2}t} + |c_1|^2 e^{+i\frac{\omega}{2}t}\right)$$



$$= |c_0|^4 + |c_0|^2|c_1|^2 e^{i\omega t} + |c_0|^2|c_1|^2 e^{-i\omega t} + |c_1|^2$$

$$= |c_0|^4 + |c_1|^4 + 2|c_0|^2|c_1|^2\cos(\omega t)$$

$$= \left(|c_0|^2 + |c_1|^2\right)^2 - 2|c_0|^2|c_1|^2 + 2|c_0|^2|c_1|^2\cos\ldots$$

$$= 1 + 2|c_0|^2|c_1|^2\left(\cos(\omega t) - 1\right) \quad //$$

1)d) $\langle e_0|e_0\rangle = \frac{1}{2}(1+1) = 1$

$\langle e_1|e_1\rangle = \frac{1}{2}(1+(-1)^2) = 1$

$\langle e_0|e_1\rangle = \frac{1}{2}(1+(-1)) = 0 = \langle e_1|e_2\rangle$

$\Rightarrow$ orthonormiert

$H|e_0\rangle = \frac{1}{\sqrt{2}}(E_0|E_0\rangle - E_1|E_1\rangle)$

$\overset{?}{=} \lambda|e_0\rangle$ nicht möglich

gleiches gilt für $H|e_1\rangle$

1)e) $|\psi\rangle = \mathbb{1}|\psi\rangle = |e_0\rangle\langle e_0|\psi\rangle + |e_1\rangle\langle e_1|\psi\rangle$

$= \frac{1}{2}(c_0+c_1)|e_0\rangle + \frac{1}{2}(c_0-c_1)|e_1\rangle$

1)f) Wir wissen $\mathbb{1} = |E_0\rangle\langle E_0| + |E_1\rangle\langle E_1| = |e_0\rangle\langle e_0| + |e_1\rangle\langle e_1|$

Ausmultiplizieren:

$|E_0\rangle\langle e_0| = \frac{1}{\sqrt{2}}|E_0\rangle\langle E_0| + \frac{1}{\sqrt{2}}|E_0\rangle\langle E_1|$ (A)

$|e_0\rangle\langle E_0| = \frac{1}{\sqrt{2}}|E_0\rangle\langle E_0| + \frac{1}{\sqrt{2}}|E_1\rangle\langle E_0|$ (B)

$|E_1\rangle\langle e_1| = \frac{1}{\sqrt{2}}|E_1\rangle\langle E_0| - \frac{1}{\sqrt{2}}|E_1\rangle\langle E_1|$ (C)

$|e_1\rangle\langle E_1| = \frac{1}{\sqrt{2}}|E_0\rangle\langle E_1| - \frac{1}{\sqrt{2}}|E_1\rangle\langle E_1|$ (D)

$\Rightarrow \frac{1}{\sqrt{2}}(A+B-C-D) = \frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{2}}|E_0\rangle\langle E_0| + \frac{2}{\sqrt{2}}|E_1\rangle\langle E_1|\right)$

$= |E_0\rangle\langle E_0| + |E_1\rangle\langle E_1| = \mathbb{1}$

2) $\psi(x) = \alpha \frac{x}{x_0} e^{-x/x_0}$

$$\psi'(x) = \frac{\alpha}{x_0} e^{-x/x_0} - \alpha \frac{x}{x_0^2} e^{-\frac{x}{x_0}}$$

$$\psi''(x) = -\frac{\alpha}{x_0^2} e^{-x/x_0} - \frac{\alpha}{x_0^2} e^{-x/x_0} + \alpha \frac{x}{x_0^3} e^{-\frac{x}{x_0}}$$

$$= \alpha \frac{x}{x_0} e^{-x/x_0} \left(-\frac{2}{x x_0} + \frac{1}{x_0^2} \right)$$

$$= \psi(x) \left(-\frac{2}{x x_0} + \frac{1}{x_0^2} \right)$$

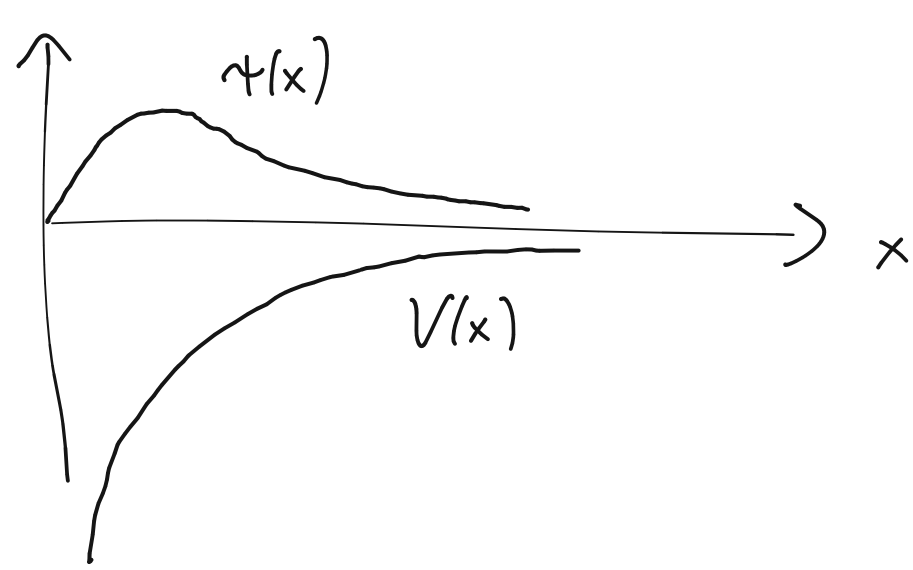
SGL.: $-\frac{\hbar^2}{2m} \psi''(x) + V(x)\psi(x) = E\psi(x)$

$$\Rightarrow -\frac{\hbar^2}{2m} \cancel{\psi(x)} \left(-\frac{2}{x x_0} + \frac{1}{x_0^2} \right) + V(x) \cancel{\psi(x)} = E \cancel{\psi(x)}$$

$$\Rightarrow \frac{\hbar^2}{m x x_0} - \frac{\hbar^2}{2m x_0^2} + V(x) = E$$

bei $x \to \infty$: $0 - \frac{\hbar^2}{2m x_0^2} + 0 = E$

$\Rightarrow E = -\frac{\hbar^2}{2m x_0^2}$ und damit $V(x) = -\frac{\hbar^2}{m x_0} \frac{1}{x}$



3) Vorweg $(BAB^{-1})^n = (BAB^{-1})(BAB^{-1}) \ldots$ n-mal

$$= BA\underbrace{B^{-1}B}_{\mathbb{1}}A\underbrace{B^{-1}B}_{\mathbb{1}}AB^{-1} \ldots BAB^{-1}$$

$$= BA\mathbb{1}A\mathbb{1} \ldots \mathbb{1}AB^{-1} = BA^nB^{-1} \quad (*)$$

$$Be^AB^{-1} = B\left(\sum_{n=0}^{\infty} \frac{1}{n!} A^n\right) B^{-1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{BA^nB^{-1}}_{(BAB^{-1})^n \text{ wegen } (*)} = \sum_{n=0}^{\infty} \frac{1}{n!} (BAB^{-1})^n$$

$$= e^{BAB^{-1}}$$