2. Test aus Statistischer Physik (Lösung)

1. (a) Wahrscheinlichkeitsdichte :

$$\rho(\vec{r}_1, \vec{p}_1, \cdots, \vec{r}_{N_1}, \vec{p}_{N_1}) = \frac{1}{Z_c} \frac{1}{N_1! h^{3N_1}} \prod_{i=1}^{N_1} \exp\left(-\beta \frac{|\vec{p}_i|^2}{2m}\right)$$

Kanonische Zustandssumme :

$$\begin{aligned} Z_c(V_1, T, N_1) &= \frac{1}{N_1! h^{3N_1}} \int e^{-\beta H} d^{3N_1} r d^{3N_1} p \\ &= \frac{1}{N_1! h^{3N_1}} V_1^{N_1} \left(\int_{-\infty}^{\infty} e^{-\beta p^2/(2m)} dp\right)^{3N_1} \\ &= \frac{1}{N_1! h^{3N_1}} V_1^{N_1} (2\pi m k_B T)^{3N_1/2} \end{aligned}$$

(b) Entropie :

$$\begin{aligned} S(V_1, T, N_1) &= -k_B \int \rho \ln\left(N_1! h^{3N_1} \rho\right) d^{3N_1} r d^{3N_1} p \\ &= -k_B \int \rho \left(-\ln Z_c - \sum_{i=1}^{N_1} \beta \frac{|\vec{p}_i|^2}{2m}\right) d^{3N_1} r d^{3N_1} p \\ &= k_B \ln Z_c - \frac{1}{T} \frac{\partial}{\partial \beta} \ln Z_c \\ &= k_B \ln\left[\frac{V_1^{N_1}}{N_1!}\right] - \frac{3}{2} N_1 k_B \ln\left[\frac{h^2}{2\pi m k_B T}\right] + \frac{3}{2} N_1 k_B \end{aligned}$$

(c) Im Limes $N \to \infty$

$$\begin{aligned} S(V_1, T, N_1) &\simeq N_1 k_B \ln V_1 - N_1 k_B \ln N_1 + N_1 k_B - \frac{3}{2} N_1 k_B \ln \lambda^2 + \frac{3}{2} N_1 k_B \\ &= N_1 k_B \ln \frac{V_1}{N_1} - 3 N_1 k_B \ln \lambda + \frac{5}{2} N_1 k_B \\ &= -N_1 k_B \ln n - 3 N_1 k_B \ln \lambda + \frac{5}{2} N_1 k_B \end{aligned}$$

$$\begin{aligned} \Delta S &= S(V, T, N) - S(V_1, T, N_1) - S(V_2, T, N_2) \\ &= -(N - N_1 - N_2) k_B \ln n - 3(N - N_1 - N_2) k_B \ln \lambda + \frac{5}{2}(N - N_1 - N_2) k_B \\ &= 0 \qquad (N = N_1 + N_2) \end{aligned}$$

(d)

$$\begin{aligned} \Delta S &= S(V, T, N_1) + S(V, T, N_2) - S(V_1, T, N_1) - S(V_2, T, N_2) \\ &= k_B \ln\left[\frac{V^{N_1}}{N_1!}\right] + k_B \ln\left[\frac{V^{N_2}}{N_2!}\right] - k_B \ln\left[\frac{V_1^{N_1}}{N_1!}\right] - k_B \ln\left[\frac{V_2^{N_2}}{N_2!}\right] \\ &= N_1 k_B \ln\left[\frac{V}{V_1}\right] + N_2 k_B \ln\left[\frac{V}{V_2}\right] > 0 \quad \text{weil } V > V_1 \text{ und } V > V_2 \end{aligned}$$

2. Zustandssumme :

$$
\begin{aligned}
Z_{\mathrm{GK}} &= \sum_{N=0}^{\infty} \frac{1}{N!h^{3N}} e^{\beta\mu N} \int \exp\left(-\beta \sum_{i=1}^{N} \frac{|\vec{p}_i|^2}{2m}\right) d^{3N}r d^{3N}p \\
&= \sum_{N=0}^{\infty} \frac{V^N}{N!h^{3N}} e^{\beta\mu N} \left(\int e^{-\beta p^2/(2m)} dp\right)^{3N} \\
&= \sum_{N=0}^{\infty} \frac{V^N}{N!h^{3N}} e^{\beta\mu N} \left(2\pi m k_B T\right)^{3N/2} \\
&= \exp\left[e^{\beta\mu} V \left(\frac{\sqrt{2\pi m k_B T}}{h}\right)^3\right]
\end{aligned}
$$

(a) Mittlere Teilchenzahl

$$
\begin{aligned}
\langle N \rangle &= \frac{1}{Z_{\mathrm{GK}}} \sum_{N=0}^{\infty} \frac{N}{N!h^{3N}} e^{\beta\mu N} \int \exp\left(-\beta \sum_{i=1}^{N} \frac{|\vec{p}_i|^2}{2m}\right) d^{3N}r d^{3N}p \\
&= \frac{1}{Z_{\mathrm{GK}}} \frac{1}{\beta} \frac{\partial}{\partial\mu} Z_{\mathrm{GK}} \\
&= \frac{1}{\beta} \frac{\partial}{\partial\mu} \ln Z_{\mathrm{GK}} \\
&= \frac{1}{\beta} \frac{\partial}{\partial\mu} \left(e^{\beta\mu} V \left(\frac{\sqrt{2\pi m k_B T}}{h}\right)^3\right) \\
&= e^{\beta\mu} V \left(\frac{\sqrt{2\pi m k_B T}}{h}\right)^3
\end{aligned}
$$

(b) Mittlere Energie

$$
\begin{aligned}
\langle E \rangle &= \frac{1}{Z_{\mathrm{GK}}} \sum_{N=0}^{\infty} \frac{H}{N!h^{3N}} e^{\beta\mu N} \int \exp\left(-\beta \sum_{i=1}^{N} \frac{|\vec{p}_i|^2}{2m}\right) d^{3N}r d^{3N}p \\
&= \frac{1}{Z_{\mathrm{GK}}} \sum_{N=0}^{\infty} \frac{H}{N!h^{3N}} z \int \exp\left(-\beta \sum_{i=1}^{N} \frac{|\vec{p}_i|^2}{2m}\right) d^{3N}r d^{3N}p \\
&= -\frac{1}{Z_{\mathrm{GK}}} \left(\frac{\partial}{\partial\beta} Z_{\mathrm{GK}}\right)_{V,z} \\
&= -\left(\frac{\partial}{\partial\beta} \ln Z_{\mathrm{GK}}\right)_{V,z} \\
&= -\left(\frac{\partial}{\partial\beta} zV \left(\frac{\sqrt{2\pi m}}{h\beta^{1/2}}\right)^3\right)_{V,z} \\
&= \frac{3}{2} k_B T z V \left(\frac{\sqrt{2\pi m k_B T}}{h}\right)^3 \\
&= \frac{3}{2} k_B T \langle N \rangle
\end{aligned}
$$

(c) Wärmekapazität

$$
C_V = \left(\frac{\partial \langle E \rangle}{\partial T}\right)_{V,\langle N \rangle} = \frac{3}{2} \langle N \rangle k_B
$$

3.
$$J = k_B T \sum_{n_x, n_y} \ln\left(1 - \exp\left[-\beta\left(\varepsilon_{n_x,n_y} - \mu\right)\right]\right)$$

mit
$$\varepsilon_{n_x,n_y} = \frac{h^2}{2mL^2}(n_x^2 + n_y^2)\,.$$

$$\langle N \rangle = -\frac{\partial}{\partial \mu} J = \sum_{n_x,n_y} \frac{1}{\exp\left[\beta\left(\varepsilon_{n_x,n_y} - \mu\right)\right] - 1}$$

$$\langle N_e \rangle = \sum_{n_x,n_y \neq 0} \frac{1}{\exp\left[\beta\left(\varepsilon_{n_x,n_y} - \mu\right)\right] - 1}$$

Lösung 1

Phasenraumvolumen (2D Gas)
$$\Phi(\varepsilon) = L^2 \int_{(p_x^2+p_y^2)/(2m)<\varepsilon} d^2p = 2\pi m L^2 \varepsilon$$

Zustandsdichte
$$D_0(\varepsilon) = \frac{1}{h^2} 2\pi m L^2$$

$$\begin{aligned}\langle N_e \rangle &= \sum_{n_x,n_y \neq 0} \frac{1}{\exp\left[\beta\left(\varepsilon_{n_x,n_y} - \mu\right)\right] - 1} \\ &= \frac{2\pi m L^2}{h^2} \int_0^\infty \frac{1}{\exp\left[\beta\left(\varepsilon - \mu\right)\right] - 1} d\varepsilon\end{aligned}$$

Lösung 2

$$\begin{aligned}\langle N_e \rangle &= \sum_{n_x,n_y \neq 0} \frac{1}{\exp\left[\beta\left(\varepsilon_{n_x,n_y} - \mu\right)\right] - 1} \\ &= \left(\frac{L}{2\pi}\right)^2 \int \frac{1}{\exp\left[\beta\left(\hbar^2(k_x^2 + k_y^2)/(2m) - \mu\right)\right] - 1} d^2k \\ &= \frac{L^2}{2\pi} \int_0^\infty \frac{1}{\exp\left[\beta\left(\hbar^2 k^2/(2m) - \mu\right)\right] - 1} k dk \\ &= \frac{mL^2}{2\pi\hbar^2} \int_0^\infty \frac{1}{\exp\left[\beta\left(\varepsilon - \mu\right)\right] - 1} d\varepsilon\end{aligned}$$