Musterlösung Nachtest QT II WS12

A1 (B3)

a) • unpolarisiert $\Rightarrow \rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

• Messung $S_x = -\frac{\hbar}{2}$ / $+\frac{\hbar}{2}$ $\Rightarrow \rho = |\downarrow\rangle_x \langle\downarrow|$ / $|\uparrow\rangle_x\langle\uparrow|_x$

In z-Basis

$S_x = \frac{\hbar}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \Rightarrow |\downarrow\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$

$|\uparrow\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$

$\Rightarrow \rho = \frac{1}{2}\begin{pmatrix} 1 & \mp 1 \\ \mp 1 & 1 \end{pmatrix}$

• Da $|\uparrow\rangle_x$ ($|\downarrow\rangle_x$) EF zu $H = -2\mu_B B_x S_x$ ändert sich ρ nicht.

b) $H_D = c\,\vec{\alpha}\vec{p} + \beta m c^2 \quad (-i\hbar \frac{\partial}{\partial \ell})$

• $[L_i, H_D] = \varepsilon_{ijk}[r_j p_k, c\,\alpha_\ell p_\ell] = c\,\varepsilon_{ijk}\, p_k \alpha_\ell \underbrace{[r_j, p_\ell]}_{-\frac{\hbar}{i}\delta_{j\ell}} = i\hbar c\,\varepsilon_{ijk}\alpha_j p_k$

$[\vec{L}^2, H_D] = L_i[L_i, H_D] + [L_i, H_D]L_i$

$= i\hbar c\,\varepsilon_{ijk}\alpha_j (L_i p_k + p_k L_i) \neq 0$

• $[\Sigma_i, \alpha_k] = \begin{pmatrix} \sigma_i & \\ & \sigma_i \end{pmatrix}\begin{pmatrix} & \sigma_k \\ \sigma_k & \end{pmatrix} - \begin{pmatrix} & \sigma_k \\ \sigma_k & \end{pmatrix}\begin{pmatrix} \sigma_i & \\ & \sigma_i \end{pmatrix} = \begin{pmatrix} & [\sigma_i, \sigma_k] \\ [\sigma_i, \sigma_k] & \end{pmatrix} = 2i\varepsilon_{ik\ell}\begin{pmatrix} & \sigma_\ell \\ \sigma_\ell & \end{pmatrix}$

$\Rightarrow [\Sigma_i, H_D] = 2i\,\varepsilon_{ik\ell}\, p_k \alpha_\ell \neq 0$

• $\Sigma^2 = \begin{pmatrix} \sigma_i & \\ & \sigma_i \end{pmatrix}\begin{pmatrix} \sigma_i & \\ & \sigma_i \end{pmatrix} = 3\begin{pmatrix} \mathbb{1} & \\ & \mathbb{1} \end{pmatrix} \Rightarrow [\Sigma^2, H_D] = 0$

c) $\{b_k, b_k^+\} = b_k b_k^+ + b_k^+ b_k = |u|^2 \underbrace{(a_k a_k^+ + a_k^+ a_k)}_{=1} + |v|^2 \underbrace{\{a^+_{-k}, a_{-k}\}}_{=1}$

$-v u^* \underbrace{(a_{-k} a_k + a_k a_{-k})}_{=0} - u v^* \underbrace{\{a_k, a^+_{-k}\}}_{=0} = 1$

$\{b_k, b_k\} = \{b_k^+, b_k^+\} = \{b_k^+, b_{-k}\} = \{b_{\pm k}, b^+_{\mp k}\} = 0$

da hier nur $\{a_k, a^+_{-k}\} = \{a_{\pm k}, a_{\pm k}\} = \{a_k^+, a_{-k}\} = \{a^+_{\pm k}, a^+_{\pm k}\} = 0$ beitragen

$\{b_k^+, b_{-k}^+\} = u\,v^* \underbrace{\{a_k^+, a_k\}}_{=1} - v u^* \underbrace{\{a_{-k}, a_{-k}^+\}}_{=1} = 0$

$\{b_k, b_{-k}\} = u^* v\,\{a_k, a_k^+\} - v^* u\,\{a_{-k}^+, a_{-k}\} = 0$

Musterlösung Bsp. 2 Streuung

$$V(r) = \frac{\alpha}{r^2}$$

a) $$f_{1.B.} = -\frac{4m\pi^2}{\hbar^2}\tilde{V}(q)$$

$$\tilde{V}(q) = \langle \vec{k}'|V|\vec{k}\rangle = \frac{1}{(2\pi)^2}\int_0^\infty dr\, r^2 \frac{\alpha}{r^2}\frac{2i}{iqr}\sin(qr) =$$

$$= \frac{\alpha}{2\pi^2}\int_0^\infty dr \frac{\sin(qr)}{qr} = \left| \begin{array}{l} qr = x \\ dr = \frac{dx}{q} \end{array} \right| = \frac{\alpha}{2\pi^2}\int_0^\infty \frac{dx}{q}\frac{\sin(x)}{x} =$$

$$= \underline{\underline{\frac{\alpha}{4\pi q}}} \qquad \Rightarrow f_{1.B.} = \underline{-\frac{m\pi\alpha}{\hbar^2 q}}$$

b) $$\frac{d\sigma}{d\Omega} = |f_{1.B.}|^2 = \frac{m^2\pi^2\alpha^2}{\hbar^4 q^2} = \underline{\frac{m^2\pi^2\alpha^2}{\hbar^4 2k^2(1-\cos\theta)}}$$

c) $$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta \frac{m^2\pi^2\alpha^2}{\hbar^4 2k^2(1-\cos\theta)} =$$

$$= \frac{m^2\pi^2\alpha^2}{\hbar^4 \cancel{2}k^2}\cancel{2}\pi \int_0^\pi d\theta \frac{\sin\theta}{1-\cos\theta} = \left| \begin{array}{l} x = \cos\theta \\ d\theta = -\frac{1}{\sin\theta}dx \end{array} \right| =$$

$$= -\frac{m^2\pi^3\alpha^2}{\hbar^4 k^2}\int_{x=1}^{-1} dx \frac{1}{1-x} =$$

$$= \frac{m^2\pi^3\alpha^2}{\hbar^4 k^2}\Big[\ln|1-x|\Big]_{x=1}^{-1} \rightarrow \underline{\underline{\infty}}$$

Potential erfüllt nicht das Konvergenzkriterium der Bornschen Reihe $V(r) \sim \frac{1}{r^\alpha}$, $\alpha > 3$

Musterlösung Bsp. 3: Sudden Approximation

1. a) $H = \frac{p^2}{2m} + V(x)$ mit $V(x) = \begin{cases} 0<x<a: & 0 \\ \text{sonst}: & \infty \end{cases}$

$\tilde{H} = \frac{p^2}{2m} + \tilde{V}(x)$ mit $\tilde{V}(x) = \begin{cases} 0<x\leq b: & 0 \\ \text{sonst}: & \infty \end{cases}$

$$\psi(x) = \sqrt{\tfrac{2}{a}}\sin\left(\tfrac{n\pi}{a}x\right)\theta(a-x)\theta(x)\ ,\quad E = \frac{\hbar^2 n^2\pi^2}{2ma^2}\ ,\ n\geq 1$$

$$\tilde{\psi}(x) = \sqrt{\tfrac{2}{b}}\sin\left(\tfrac{n\pi}{b}x\right)\theta(b-x)\theta(x)\ ,\quad \tilde{E} = \frac{\hbar^2 n^2\pi^2}{2mb^2}\ ,\ n\geq 1$$

b) Sudden approximation anwendbar, falls bspw.

$$\boxed{\tau\cdot\omega_{k0} \ll 1} \quad \text{mit} \quad \omega_{k0} = \frac{\hbar k^2\pi^2}{ma^2}$$

c) $P_{n\leftarrow 0} = |\langle\tilde{\psi}_n|\psi_0\rangle|^2$

$$\langle\tilde{\psi}_n|\psi_0\rangle = \frac{2}{\sqrt{ab}}\int_0^a dx\, \sin\left(\tfrac{n\pi}{b}x\right)\sin\left(\tfrac{\pi}{a}x\right) \quad (\text{weil } a<b)$$

$$\int dx\ldots = -\frac{1}{4}\int_0^a dx\left[e^{i\pi x\left(\frac{n}{b}+\frac{1}{a}\right)} - e^{i\pi x\left(\frac{n}{b}-\frac{1}{a}\right)} - e^{-i\pi x\left(\frac{n}{b}-\frac{1}{a}\right)} + e^{-i\pi x\left(\frac{n}{b}+\frac{1}{a}\right)}\right] =$$

$$= -\int_0^a dx\left[\cos\left(\pi\left(\tfrac{n}{b}+\tfrac{1}{a}\right)x\right) - \cos\left(\pi\left(\tfrac{n}{b}-\tfrac{1}{a}\right)x\right)\right] =$$

$$= -\frac{ab}{\pi}\left[\frac{\sin\left(\pi\left(\frac{n}{b}+\frac{1}{a}\right)a\right)(an-b) - \sin\left(\pi\left(\frac{n}{b}-\frac{1}{a}\right)a\right)(an+b)}{(an+b)(an-b)}\right] =$$

$$= -\frac{ab}{\pi}\left[\frac{an\left(\sin\left(\pi\left(\frac{n}{b}+\frac{1}{a}\right)a\right) - \sin\left(\pi\left(\frac{n}{b}-\frac{1}{a}\right)a\right)\right) - b\left(\sin\ldots+\sin\ldots\right)}{a^2n^2-b^2}\right] =$$

$$= \left|\text{Summensätze}\right| = -\frac{ab^2\sin\left(\frac{n\pi a}{b}\right)}{2\pi(a^2n^2-b^2)} \Rightarrow \boxed{P_{n\leftarrow 0} = \frac{4ab^3\sin^2\left(\frac{n\pi a}{b}\right)}{\pi^2(a^2n^2-b^2)^2}}$$

Beispiel 4 | Gruppe B Beispiel 1

a) $\psi(r_1, r_2) = N\left[\psi_{1(\alpha)}(r_1)\psi_{2(\beta)}(r_2) + \psi_{2(\beta)}(r_1)\psi_{1(\alpha)}(r_2)\right]$ mit $N = \sqrt{\frac{1}{2(1+|S|^2)}}$

$N^2 \int dr_1 dr_2 \, |\psi(r_1, r_2)|^2 = 1 \Rightarrow N^2 = \left[2 + 2\left|\underbrace{\int dr\, \psi_1^*(r)\psi_2(r)}_{S = \text{Überlapp Integral}}\right|^2\right]$

b) $\hat{n}(r) = \sum_i \delta(r - r_i)$

$n(r) = \langle\psi|\hat{n}(r)|\psi\rangle = \int dr_1 dr_2\, \psi^*(r_1, r_2)\left[\delta(r-r_1) + \delta(r-r_2)\right]\psi(r_1, r_2) =$

$= 2N^2\left[|\psi_1(r)|^2 + |\psi_2(r)|^2 + Re\left[\psi_1(r)\psi_2^*(r) S\right]\right] =$

Wenn die Teilchen unterschiedlich wäre, hätte man $\psi_{dist}(r_1, r_2) = \psi_1(r_1)\psi_2(r_2)$ (oder, andersherum, $\psi_2(r_1)\psi_1(r_2)$), d.h.

$n_{dist}(r) = \langle\psi_{dist}|\hat{n}(r)|\psi_{dist}\rangle = |\psi_1(r)|^2 + |\psi_2(r)|^2$

Vergleich und Interpretation: Für r, wo $\psi_1^*(r)\psi_2(r) \approx \emptyset$,

$n(r) < n_{dist}(r)$ weil $2N^2 = \frac{1}{1+|S|^2} < 1$

Aber, dann, für r, wo $\psi_1^*(r)\psi_2(r) \neq \emptyset$, muss

$n(r) > n_{dist}(r)$, da die Teilchenzahl $\int dr\, n(r) = 2$ für die beide Fälle die gleiche ist.

c) Änderung für FERMIONEN (zwei mit $S = \frac{1}{2}$ und $S_z = +\frac{\hbar}{2} \Rightarrow \uparrow$)

$\psi(r_1, r_2) = N_F\left(\psi_{1,\uparrow}(r_1)\psi_{2,\uparrow}(r_2) - \psi_{2,\uparrow}(r_1)\psi_{1,\uparrow}(r_2)\right)$ mit $N_F = \sqrt{\frac{1}{2(1-|S|^2)}}$

$n(r) = 2N_F^2\left[|\psi_1(r)|^2 + |\psi_2(r)|^2 - 2Re\left[\psi_1(r)\psi_2^* S\right]\right]$

und jetzt für r, wo $\psi_1^*(r)\psi_2^*(r) \approx 0$ $\quad n(r) > n_{dist}(r)$

und für r, wo $\psi_1(r)\psi_2(r) \neq 0$ $\quad n(r) < n_{dist}(r)$